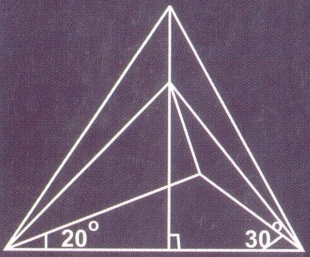
GEMS

from

THE MATHEMATICS TEACHER

JUNIOR

V.SESHAN



$$a^2 + b^2 + c^2 = \sqrt{3}abc$$

$$\frac{1}{x} - \frac{1}{xy} - \frac{1}{xyz} = \frac{19}{97}$$

THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

GEMS

FROM

THE MATHEMATICS TEACHER

(JUNIOR LEVEL)

Compiled and Edited

by

V. SESHAN

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Between Us

Dear Reader,

It is matter of pleasure and satisfaction that one of the promised books has been ready for use now. The earlier book titled GEMS from the Mathematics Teacher has been popular which contained selected questions and solutions of all levels from our journal up to 1995. There was a demand for a similar one beyond that year with specified group –Junior, Inter and the like. Accordingly we planned one GEMS-Junior and one GEMS –Inter and the former is now ready.

The problems were chosen, checked for the solutions and additional information as hints to plan solving such problems by Sri V.Seshan, who has a background of Mathematics teaching for the past 5 decades, besides directly associated with the INMO culture as Regional Coordinator in Bombay. He is also in demand from several institutions including the Kendriya Vidyalaya Sangathan, to guide their teachers and students for Olympiad culture. I take this opportunity to thank him on behalf of the AMTI for this book production, preparing the manuscripts, correcting proofs and editing the same.

The prospective users will find the material useful for NMTC, Olympiads, IIT and the like besides spending leisure time in problem solving. We hope and trust that the well wishers of The AMTI will make use of this material also. Suggestions for improvement are always welcome as ultimately quality products will stand the tests of time.

With kind regards/best wishes,

Yours sincerely,

(M.MAHADEVAN)

PREFACE

This book ***'Gems from the Mathematics Teacher'*** is an extremely useful compilation of problems set for various Olympiads organised by the Association of Mathematics Teachers of India and the National Board for Higher Mathematics (of Govt. of India) for the last ten years. Useful theorems, formulae, rules, ideas are added.

Efforts have been made

(i) to simplify solutions of some problems

(ii) to give alternate solutions for some problems depending upon the '*complexities*' of the problem / solution.

Although this book is meant for competitors at Junior Level, set, a lot of problems, especially in Geometry (school syllabus only) have been added with the view to give a wider exposure in problem solving strategies at slightly higher level also.

This book is recommended for

(i) beginners who will be motivated towards developing better mathematical abilities.

(ii) Advanced / talented students who want to deepen their interest in the subject and develop creative faculties in problem solving.

(iii) Competitors (of Mathematical Olympiads) and Talent Nature Contests at various levels, organised by Professional organizations such as the National Board for Higher Mathematics (RMO / GMO / INMO). The Association of Mathematics Teachers of India (AMTI) and the like.

Teachers, who desire to improve upon and further their skills and knowledge in problem solving as well as the subject and inculcate the same for teaching / learning activities as well as for Guidance to motivated students to prepare for the various Olympiads, will find this book a very valuable asset.

Suggestions for the improvement of this book are very welcome.

V.Seshan

- *Presidential Awardee (1987)*
- *Fulbright Teacher Awardee (USA-1970)*
- *Pee Vee National Awardee (2000)*
- *International Order of Merit (BC-UK-2000)*
- *Advisor, National Science Olympiad Foundation (Since 1989)*
- *Advisor, American Biographical Institute (USA-2000)*
- *Rotary (Int.) Awardee (1992*
- *Senior Principal AEES (Govt. of India) (1987-1995)*
- *Director, Bharatiya Vidya Bhavan, Baroda (1995-2002)*
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SOME HINTS USEFUL FOR PROBLEMS SOLVING AT OLYMPIAD (JUNIOR LEVEL)

(1) Identities:

(a) If $a+b+c=0$, $a^3+b^3+c^3=3abc$

(b) If $a+b+c=0$, $a^4+b^4+c^4=\frac{1}{2}(a^2+b^2+c^2)^2$

(2) Periodic Function:

A function f is said to be periodic, with period k, if $f(x+k)=f(x)$ for all x.

(3) Pigeon Hole Principle:

If more than n objects are distributed in n boxes, then, at least, one box, has more than one object in it.

(4) Polynomial Equations:

(a) Every polynomial equation of degree $n(\geq 1)$ has exactly n roots.

(b) If a polynomial equation with *real coefficients* has a *complex root* $p+iq (p, q \in R,\ q \neq 0,\ i^2=-1)$ then, it also has a complex root $p-iq$.

(c) If a polynomial equation with *rational coefficients* has an *irrational root* $p+\sqrt{q}$ (p, q rational, $q>0$, q not the square of any rational number), then it also has an irrational root $p-\sqrt{q}$.

(d) If the rational number $\frac{p}{q}$ (a fraction in its lowest terms so that p, q are integers, prime to each other, $q \neq 0$), is a root of the equation

$a_0\, x^n + a_1\, x^{n-1} + \ldots + a_n = 0$

where $a_0, a_1, a_2, \ldots, a_n$ are integers and $a_n \neq 0$, then, p *is a divisor of* a_n *and* q *is a divisor of* a_0.

(e) A number α is a *common root* of the polynomial equations $f(x) = 0$ and $g(x) = 0$ if and only if it is a root of $h(x) = 0$ where $h(x)$ is the G.C.D of $f(x)$ and $g(x)$.

(f) A number α is a *repeated root* of a polynomial equation of $f(x) = 0$ if and only if it is a common root of $f'(x) = 0$ and $f(x) = 0$.

(g) If α, β, γ are the roots of the equation

$ax^3 + bx^2 + cx + d = 0$, then the following relations hold:

(i) $\alpha + \beta + \gamma = \frac{-b}{a}$

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

(iii) $\alpha\beta\gamma = \frac{-d}{a}$

(h) If $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$ax^4 + bx^3 + cx^2 + dx + e = 0$, then,

(i) $\alpha + \beta + \gamma + \delta = \frac{-b}{a}$ (i.e $\sigma\alpha = \frac{-b}{a}$)

(ii) $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$ (i.e. $\sigma\alpha\beta = \frac{c}{a}$)

(iii) $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a}$

(iv) $\alpha\beta\gamma = \frac{e}{a}$.

(i) An equation containing (involving) an *unknown function* is called a *functional equation.*

(j) **The greatest integer function:**

[ ] is defined by setting $[x]$ = *greatest integer not exceeding* x for every real x.

(5) Linearity Property:

If a/b and a/c, then $a/(pb+qc)$ (i.e., if a divides b and a divides c, then a divides $(pb+qc)$).

(6) Euclid's Algorithm:

Let a and b be two non-zero integers. Then (a,b) exists and is unique. Also there exists m and n such that

$(a,b) = am + bn$ (Note: (a,b) means g.c.d of (a,b)).

(7) Congruencies:

Let a, b, m be integers, $m > 0$. Then, we say that, a is congruent to b modulo m if $m|(a-b)$. We denote this by $a \equiv b(\text{mod } m)$.

(8) Let $a \equiv b \pmod m$; $c \equiv d \pmod m$. Then

(i) $a + c \equiv b + d \pmod m$

(ii) $a - c \equiv b - d \pmod m$

(iii) $ac \equiv bd \pmod m$

(iv) $pa + qc \equiv pb + qd \pmod m$ for all integers p and q

(v) $a^n \equiv b^n \pmod m$ for all positive integers n

(vi) $f(a) \equiv f(b) \pmod m$ for every polynomial f with integer coefficients.

(9) An integer x_0 satisfying the linear congruence

$$ax_0 \equiv b(\text{mod} m)$$

has a solution. Furthermore, if x_0 is a solution, then, the set of all solutions is precisely $(x_0 + km : k \epsilon z)$.

(10) Let N be a positive integer greater than 1, say, $N = a^p b^q c^r \ldots$ where a, b, c are positive integers.

The number of ways in which N can be resolved into 2 factors is $\frac{1}{2}(p+1)(q+1)(r+1)\ldots$.

(11) Number of ways in which a composite number can be resolved into two factors, which are prime to each other! is 2^{n-1}, where n is the number of *distinct prime factors* in the expansion for N.

(12) Let N be a positive integer greater than 1 and let $N = a^p \cdot b^q \cdot c^r \ldots$ where a, b, c are distinct primes and $p, q, r, \ldots$ positive integers. Then the sum of all the divisors in the product is equal to

$$\frac{a^{p+1}-1}{a-1} \cdot \frac{b^{q+1}-1}{b-1} \cdot \frac{c^{r+1}-1}{c-1} \ldots$$

(13) The highest power of prime p which is contained in $\angle n$ is $[\frac{n}{p}] + [\frac{n}{p^2}] + [\frac{n}{p^3}] + \ldots$ where $[\cdot]$ denotes the greatest integer function.

(14) If m_a, m_b, m_c are the lengths of the medians of a triangle on the sides a, b, c, then

(i) $m_a + m_b + m_c < a + b + c$

(ii) $3(a + b + c) < 4(m_a + m_b + m_c)$

(15) The orthocenter of the original triangle is the **incentre** of its pedal triangle. ($\triangle$ formed by attitudes of the given $\triangle$).

(16) In any triangle, the circumventer (S), the centroid (G), and orthocenter (H) are collinear. This line (SGH) is called Euler's line. Further, $SG : GH = 1 : 2$.

(17) The circle through the midpoints of the sides of a triangle passes through the feet of the perpendiculars from the vertices as well as the mid points of the segments joining the orthocentre to its vertices. This circle is called the Mid point circle on the Nine point circle. Its center is the midpoint of the line segment joining the orthocentre and the circumcentre (i.e., SH)
Note: S-N-H. The radius of this circle is one-half of the circum radius (i.e $\frac{R}{2}$). The Euler's line (SGH) passes through the center of the nine point circle (i.e S-G-N-H).

(18) Equilateral triangle BCP, CAQ, ABR are constructed externally on the sides BC,CA and AB of $\triangle ABC$. Then AP=BQ=CR.

(19) Two external bisectors and third internal bisector of the angles of a triangle are *concurrent.* The point of concurrence is an *excentre* of the triangle. There are three encentres for a triangle.

(20) In $\triangle ABC, AC > AB$. AD is the angle bisector of $\angle A$ and H is its orthocentre.
Then $\angle DAH = \frac{1}{2}(\angle B \sim \angle C)$.

(21) In $\triangle ABC, \angle A = 45°$ and H is its orthocentre. Then $AH = BC$.

(22) If, in a triangle, one angle is $30°$, its opposite side is equal to its circumradius in length.

(23) If the diagonals of a cyclic quadrilateral intersect at right angles, the line drawn through their point of intersection and perpendicular to a side, bisects the opposite side.

(24) If $ABCD$ is a cyclic quadrilateral, then
$AB \cdot CD + AD \cdot BC = AC \cdot BD$ (Ptolemy's Theorem)

(25) In $\triangle ABC$, S is the circumcentre, H is its orthocentre and D is the mid point of BC.
Then $AH = 2SD$.

(26) If, from a point P on the circumcircle of $\triangle ABC$, perpendiculars are drawn to the sides of $\triangle ABC$, then, the feet of the perpendiculars are collinear (pedal line).

(27) In $\triangle ABC$, the altitude AD from the vertex A to the opposite side BC, is produced to meet the cicumcircle of the triangle at Q and if H is the orthocentre of the triangle, then $HD = DQ$.

(28) If AE is a diameter of the circumcircle of $\triangle ABC$ and AD is drawn perpendicular to BC, then

$$AB \cdot AC = AD \cdot AE.$$

(29) The vertical angle A of $\triangle ABC$ is bisected internally by a straight line cutting the base BC at D.
Then $AB \cdot AC = AD^2 + BD \cdot DC$.

(30) The radical axes of three non-concentric circles, taken in pairs, are either all parallel or concurrent.

(31) The radical axis of two non-concentric circles is a straight line perpendicular to the line of centers.

(32) (a) If a transversal cuts the sides BC, CA, AB of ΔABC in D, E, F respectively, then

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=-1. \quad \text{(Menelau's Theorem)}$$

(b) If D, E, F are three points on each of the sides BC, CA, AB of ΔABC or on their extension, such that,

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=-1,$$

then, D, E, F are collinear.

(33) (a) If three concurrent lines are drawn through vertices A, B, C of ΔABC to meet the opposite sides in D, E, F respectively, then

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=+1. \quad \text{(Ceva's Theorem)}$$

(b) If three lines AD, BE, CF satisfy the condition that

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=+1,$$

then they are concurrent.

Note: The concurrent lines AD, BE, CF are called Cevians.

(34) In ΔABC, if AD is the internal bisector of $\angle A$. Then $\frac{BD}{DC}=\frac{AB}{AC}$. If AD' is the external bisector of $\angle A$ meeting BC at D', then $\frac{BD'}{D'C}=\frac{AB}{AC}$.

Conversely, if D and D' are points on BC of ΔABC such that $\frac{BD}{DC}=\frac{BD'}{D'C}=\frac{AB}{AC}$, then, D and D' represent the extremities of internal and external bisectors of $\angle A$.

(35) In any $\Delta ABC, r=(s-a)\tan(A/2)$, where

$\angle A = 90^\circ$; $abc = 4R\Delta$; $\Delta = rs$

(36) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ (Sine rule)

$$\left.\begin{aligned} \cos A &= \frac{b^2+c^2-a^2}{2bc}\,; \\ \cos B &= \frac{c^2+a^2-b^2}{2ca}\,; \\ \cos C &= \frac{a^2+b^2-c^2}{2ab}\,; \end{aligned}\right\} \quad \text{(cosine rule)}$$

(37) $\sin(A/2) = \sqrt{\frac{(s-b)(s-c)}{bc}}\,;$

$$\sin(B/2) = \sqrt{\frac{(s-c)(s-a)}{ca}}\,;$$

$$\sin(C/2) = \sqrt{\frac{(s-a)(s-b)}{ab}}\,;$$

$$\cos(A/2) = \sqrt{\frac{s(s-a)}{bc}}\,;$$

$$\cos(B/2) = \sqrt{\frac{s(s-b)}{ca}}\,;$$

$$\cos(C/2) = \sqrt{\frac{s(s-c)}{ab}}\,;$$

$$\tan(A/2) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\,;$$

$$\tan(B/2) = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}\,;$$

$$\tan(C/2) = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\,.$$

(38) $M = 4R\sin(A/2)\cdot\sin(B/2)\cdot\sin(C/2)\,.$

(39) $\triangle = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B\,.$

(40) $(\sqrt{a}-\sqrt{b})^2 \geq 0 \Rightarrow \frac{a+b}{2} \geq \sqrt{ab}$
i.e., Arithmetic mean $\geq$ Geometric mean $(A.M \geq G.M)$.

(41) Root Mean Square (RMS $\geq$ A.M) Arithmetic mean

$$\text{i.e.,} \qquad \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2} \Rightarrow \frac{(a-b)^2}{2} \geq 0.$$

(42) If x is *real* and $Ax^2+Bx+C \leftrightarrow B^2-4AC \geq 0$.

(43) If $A>0$ and x is real, then $Ax^2+Bx+C \geq 0$
$\Rightarrow 4AC-B^2 \geq 0$.

(44) $|A|-|B| \leq |A+B| \leq |A|+|B|$.

(45) Triangular Inequality: The lengths a,b,c can represent the sides of a triangle if and only if $a+b>c$, $b+c>a$, $c+a>b$ are met simultaneously.

(46) In any $\triangle ABC$, if a,b,c represent the sides of a triangle, then, $a>|c-b|$, $b>|a-c|$, $c>|b-a|$.

(47)
The area of the triangle $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$, will be real. Thus 'reality' of the area $(\triangle)$ formed by the lengths a,b,c is the necessary and sufficient condition for the existence of the triangle.

(48) Mean Inequalities:

(i) RMS $\geq$ A.M $\geq$ G.M $\geq$ H.M

(ii) $$\sqrt{\frac{a^2+b^2+c^2}{3}} \geq \left(\frac{a+b+c}{3}\right)$$

(iii) $$\frac{a+b+c}{3} \geq \left(\frac{a^{-1}+b^{-1}+c^{-1}}{3}\right)^{-1}.$$

(49) Jenson's Inequality: If a real function $f(x)$ is positive and curves towards x-axis, then
$$\left(\frac{x+y}{2}\right) \geq \left(\frac{f(x)+f(y)}{2}\right).$$

(50) Weirstrass' Inequality:
For positive numbers $a_1, a_2, \ldots, a_n$,

$$(1+a_1)(1+a_2)\ldots(1+a_n) > 1 + a_1 + a_2 + \ldots + a_n\,.$$

If a_i are fractions (< 1), then

$$(1-a_1)(1-a_2)\ldots(1-a_n) \leq 1 - (a_1 + a_2 + \ldots + a_n)\,.$$

(51) Cauchy Schwartz Inequality:
$$(a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n)^2 \\ \leq (a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2) \quad (b_1^2 + b_2^2 + b_3^2 + \ldots + b_n^2)$$

(52) Tchebychev's Inequality:

If $x_1 \geq x_2 \geq x_3 \ldots x_n$ and $y_1 \geq y_2 \geq y_3 \ldots y_n$ or

$x_1 \leq x_2 \leq x_3 \ldots x_n$ and $y_1 \leq y_2 \leq y_3 \ldots y_n$, so to say both the sequences are either increasing or decreasing type, then
$$\frac{x_1y_1 + x_2y_2 + \ldots x_ny_n}{n} \geq \left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)\left(\frac{y_1 + y_2 + \ldots + y_n}{n}\right)$$
If one of the sequence is increasing and the other decreasing type, then, the direction of the inequality changes.

(53) If $a_i = \{a_1, a_2 \ldots a_n\}$ are positive numbers

and $b = \{b_i\}$ are various permutations of a_i, then,
$$\sum_{i=1}^{n} a_i \cdot a_i \geq \sum_{i=1}^{n} a_i b_i.$$

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UNIT 1: ALGEBRA

1.01: A man drives from his house to the station. If he drives at the rate of 10Kms per hour, he reaches the station at 6p.m. If he drives at 15Kms per hour, he would reach the station at 4p.m. At what speed, in kilometers per hour, should drive so as to reach the station at 5p.m.?

(AMTI-98)J

1.02: Find the smallest positive integer on such that

$\sqrt{n} - \sqrt{n-1} < 0.01$. (AMTI-98)J

1.03: In a farm, both men and women were working. Exactly one- third of the staff brought one child each. One day, each male employee planted 13 trees and each women employee planted 10 trees and each child planted 6 trees. A total of 159 trees were planted on that day. How many women employees were there in that farm? (AMTI-98)J

1.04: For a real number x and a positive number n, define $\binom{x}{n} = \frac{x(x-1)(x-2)\cdots(x-(n-1))}{1\cdot 2\cdot 3\cdots n}$. What is $\binom{\frac{-1}{2}}{100} \div \binom{\frac{1}{2}}{100}$?

(AMTI-98)J

1.05: In racing over a given distance d at uniform speed, A can beat B by 30 meters, B can beat C by 20 meters and A can beat C by 48 meters. Find 'd' in meters.

(AMTI-98)J

1.06: Let $x = 0.123456789101112\cdots 998999$ where the digits are obtained by writing the integers 1 through 999 in order. The 1983^{rd} digit to the right of the decimal point is q. Find q. (AMTI-98)J

1.07: For what values of 'a', the equations

$1998x^2 + ax + 8991 = 0$ and $8991x^2 + ax + 1998 = 0$ have a common root? (AMTI-98)J

1.08: Show that there are no integers x, y satisfying the equation $2x^4 + 1987 = 3y^4$. (AMTI-98)J

1.09: Find positive integers x, y, z such that $x < y < z$ and $\frac{1}{x} - \frac{1}{xy} - \frac{1}{xyz} = \frac{19}{97}$. (AMTI-98)J

1.10: Find x, y, x satisfying the equations

$$(x + y)(x + y + z) = 66$$

$$(y + z)(x + y + z) = 99$$

$$(z + x)(x + y + z) = 77$$

(AMTI-98F)J

1.11: Let $N_k = 131313\cdots131$ be the $(2k + 1)$-digit number in base 10 formed by k pieces of 13 and appended by 1 at the end. Prove that N_k is not divisible by 31 for any value of $k = 1, 2, 3\cdots$ (AMTI-98F)J

1.12: We have 12 rods each of 13 units length. They are to be put into pieces measuring 3,4,5 units so that the resulting pieces can be assembled into 13 triangles of sides 3,4,5 units. How should the rods be cut? (AMTI-98F)J

1.13: a, b, c are positive integers satisfying the equations $5a+5b+2ab = 92$, $5b+5c+2bc = 136$, $5c+5a+2ca = 244$. Find $7a + 8b + 9c$. (AMTI-98F)J

1.14: a, b, c, d are positive integers such that $a^5 = b^4$, $c^3 = d^2$ and $c - a = 19$. Find $d - b$. (AMTI-98F)J

1.15: The pages of a book are numbered 1 through n. When the page numbers of the book were added,one of the page numbers was mistakenly added twice resulting in the incorrect sum 1998. What was the number of the page that was added twice? (AMTI-98F)J

1.16: Find the least L.C.M of 20 natural numbers not necessarily different, whose sum is 801. (RMO,1998)

1.17: Given the 7-element set $A = \{a, b, c, d, e, f, g, \}$. Find a collection T of 3-element subsets of A such that each pair of elements from A occurs exactly in one of the subsets of T. (RMO-1998)

1.18: In a village 1998 persons volunteered to clean up for a fair, a rectangular field with integer sides and perimeter equal to 3996 feet. For the purpose, the field was divided into 1988 equal parts.If each part had an integer area (measured in square feet), find the length and breadth of the field. (INMO-1999)

1.19: Show that there do not exist polynomials $p(x)$ nd $q(x)$, each having integer coefficients and of degree greater than or equal to 1 such that $p(x) \cdot q(x) = x^5 + 2x + 1$ (INMO-1999)

1.20: Given any four positive distinct real numbers, show that, one can choose 3 numbers A, B, C from among them such that, all the three quadratic equations have only real roots or all the three equations have only imaginary roots, where the equations are $Bx^2 + x + C = 0$, $Cx^2 + x + A = 0$, $Ax^2 + x + B = 0$. (INMO-1999)

1.21: For which positive integer values of n, the set $\{1, 2, 3, 4, \cdots 4n\}$ can be split into n distinct 4 elements subsets $\{a, b, c, d\}$ such that $a = \dfrac{b+c+d}{3}$ (INMO-1999)

1.22: A secretary has 4 letters addressed to four different persons with envelopes bearing the addresses of these persons. Find the number of ways the careless secretary can put the letters in the envelopes so that no person gets the right envelope. (AMTI-99)

1.23: A, B, C are finite sets. A has twice as many elements as B. B has more elements than C. The number of subsets of B is 15 more than that of C. Then the number by which the number of subsets of A exceeds the number of subsets of B is n. Find this n.

(AMTI-1999)

1.24: If $x^2 + x + 1 = 0$, find the value of $x^{1999} + x^{2000}$.

(AMTI-1999)

1.25: If the common points of the graphs of $y = x^2 - 6x + 2$ and $x + 2y = 4$ are A and B find the coefficients of the midpoint of AB. (AMTI-1999)

1.26: If $[x]$ denotes the largest integer less than or equal to x, then

$$\left[\tfrac{1}{2}\right] + \left[\tfrac{1}{2} + \tfrac{1}{100}\right] + \left[\tfrac{1}{2} + \tfrac{2}{100}\right] + \left[\tfrac{1}{2} + \tfrac{3}{100}\right] + \cdots \left[\tfrac{1}{2} + \tfrac{61}{100}\right]$$

is an integer. Find that integer. (AMTI-1999)

1.27: 3 men and 5 boys can do $\frac{19}{20}$ of a work in 3 days. 4 men and 18 boys can do $\frac{14}{15}$ of the work in two days. How many days does a boy to finish the whole work?

(AMTI-1999)

1.28: If x and y are natural numbers, find the number pairs (x, y) for which $x^2 - y^2 = 31$. (AMTI-1999)

1.29: A small of tribals have a language all of whose words can have at the most 4 letters while they have only 4 letters in the alphabet of the language.

Find the number of words in the language at the most. (AMTI-1999)

1.30: In a long jump qualifying competition, the average jump length of successful competitions was 6.5m; the average jump length of unsuccessful competitions was 4.5m and the average jump length of all the competitions was 4.9m. Find the percentage of successful competitions. (AMTI-1999)

1.31: If $x + y = 5xy$, $y + z = 6yz$, $z + x = 7zx$ find the value of $x + y + z$. (AMTI-1999)

1.32: If x and y can take only natural number values, find the number of (x, y) pairs satisfying the equation $2x + 5y = 100$. (AMTI-1999)

1.33: Find the largest among the numbers and arrange them in ascending order: $(1.001)^{1000}$, $(1.01)^{1000}$, 1000, $2^9, 3^6$. (AMTI-1999)

1.34: Find the minimum number of unit squares needed to cover an equilateral triangle with side length as 2 units. (AMTI-1999)

1.35: Prove that, there are infinitely many triplets (x, y, z) of positive integers such that $x^3 + y^5 = z^7$. (AMTI-1999F)

1.36: Seven points are placed inside a square of side 1. Prove that, at least two of then, are at a distance of not greater than $\sqrt{\frac{13}{6}}$. (AMTI-1999F)

1.37: a, b, c, d are numbers of which at last one is non-zero such that $a+b+c+d=0$. Let $P = ab+bc+cd$ and $Q = ac+ad+bd$.

Prove that at least one of $19P+99Q$ and $19Q+99P$ must be negative. (AMTI-1999)

1.38: Find the smallest multiple of 15 such that each digit of the multiple is either 0 or 8. (AMTI-1999)

1.39: If p, q, r are the roots of the cubic equation

$x^3-3px^2+3q^2x-r^3=0$, prove that $p=q=r$ (RMO-99)

1.40: Find all solutions in integers m, n of the equation $(m-n)^2 = \dfrac{4mn}{m+n-1}$. (RMO-99)

1.41: If a, b, c, x are real numbers such that $abc \neq 0$ and $\dfrac{bx+(1-x)c}{a} = \dfrac{cx+(1-x)a}{b} = \dfrac{ax+(1-x)b}{c}$ then, prove that, $a+b+c=0$ or $a=b=c$. (INMO 2000)

1.42: If $\underbrace{a^n+a^n+\cdots+a^n}_{m \text{ times}} = a^{n+1}$

and $\underbrace{b^m+b^m+\cdots+b^m}_{m \text{ times}} = b^{m+1}$.

Prove that $mn-(ab-1)=1$ (AMTI-2000)

1.43: If a, b, x, y are natural numbers such that

$a^2+b^2 = 25$ and $x^2+y^2 = 13$, then, show that $(ax+by)+(ay+bx)$ has exactly one value. (AMTI-2000)

1.44: Consider the sequence of natural numbers $1, 1, 2, 2, 3, 3, 4, 4, \cdots r, r, (r+1), (r+1) \cdots$ $f(n)$ is the sum to n terms of this sequence. If K is an odd number, find $f(k^2)$. (AMTI-2000)

1.45: If $n!$ has 4 zeroes at the end and $(n+1)! = n!(n+1)$ has six zeroes at the end, find the value of n. (AMTI-2000)

1.46: Given $2x^2 + 3y^2 = 35$. Find the number of ordered pairs (x, y) where (x, y) are integers satisfying the given condition. (AMTI-2000)

1.47: Four positive integers are given. Select any three of these integers and find their A.M. (i.e. average) and to this result add the fourth number. If the four answers thus got are 17,21,23 and 29, then find the original numbers.

(AMTI-2000)

1.48: A bag contains balls of 7 different colors. The least number of balls that should be collected. So that two of them may be of the same color is x find x. Generalize your result. (AMTI-2000)

1.49: Find the least value of a for which the sum of the squares of the roots of $x^2 + (a-2)x + (1-a) = 0$ is least.

(AMTI-2000)

1.50: α and β are two numbers such that $\alpha + \beta = 6$, $\alpha \sim \beta = 8$. Then trace the equation whose roots are α and β. (AMTI-2000)

1.51: If $f(a) = a-2$ and $F(a, b) = b^2+a$, find $F(3, f(4))$.

(AMTI-2000)

1.52: a, b, c are positive integers less than 10. If $(10a+b)(10a+c) = 100a(a+1)+bc$, prove that $b+c = 10$.

1.53: x, y, z are real variables such that x varies as the cube of y and y varies as the fifth root of z. Then x varies as the nth root of z. What is the value of n?

(AMTI-2000)

1.54: Multiply the consecutive positive integers until the product $2 \cdot 4 \cdot 6 \cdot 8 \cdots$ becomes divisible by 2001. Find the largest even integer we use to satisfy this condition.

(AMTI-2000:F)

1.55: If a $a679b$ is a 5-digit number in base 10 and is divisible by 72, find the number. (AMTI-2000)

1.56: In the year 2000, I will be old as the sum of the digits of my birth year. When was I born? (AMTI-2000:F)

1.57: Prove that, if $a, b(a > b)$ are prime numbers, each containing at lest 2 digits, then $a^4 - b^4$ is divisible by 240. Also prove that, 240 is the greatest common divisor of all numbers which arise in this way. (AMTI-2000:F)

1.58: Let $f(x) = \frac{16^x}{16^x+4}$.

Evaluate the sum $\frac{1}{2000} + \frac{2}{2000} + \cdots + \frac{1999}{2000}$ (AMTI-2000:F)

1.59: Arrange the numbers from 1 to 100 as a sequence such that any 11 terms in it (not necessarily consecutive) do not form an increasing or decreasing sequence.

(AMTI-2000:F)

1.60: Find all integer solutions of the equation

$$x^2 + y^2 + z^2 = 2xyz$$ (AMTI-2000:F)

1.61: Two students from class X and several students from XI participated in a chess tournament.Each participant played once with every other. In each game, the winner has received 1 point, the loser zero and for a drawn game, both players got $\frac{1}{2}$. The two students from class X together scored 8 points and the scores of all participants in class XI are equal. How many students from class XI participated in the tournament? (AMTI-2000:F)

1.62: Consider the following sequences of natural numbers.
$S_0 : 1, 1$
$S_1 : 1, 2, 1$
$S_2 : 1, 3, 2, 3, 1$
$S_3 : 1, 4, 3, 5, 2, 5, 3, 4, 1$ and so on. S_n is formed out of s_{n-1} as follows: Between any two terms a and b in S_{n-1}, insert the sum $a+b$. The new terms together with those of s_n constitute S_n. Consider S_{100}. How many terms are there in S_{100}? How many times the term 20 occurs in S_{100}? (AMTI-2000:F)

1.63: Prove that, if n is a non-negative integer, then,it can be uniquely represented in the form
$$n = \frac{(x+y)^2 + 3x + y}{2}$$
where x, y are non-negative integers. (AMTI-2000:F)

1.64: Find all real solutions of the system of equations: $x + y = 2; xy - z^2 = 1$ (AMTI-2000:F)

1.65: Let $p(x)$ be a polynomial with integer coefficients. Prove that, if $p(x) = 7$ for four distinct integral values of x, then, $p(x) \neq 14$ for any integral value of x.
(AMTI-2000:F)

1.66: Let $ABCD$ be a square of side length 20. Let T_i $(1 = 1, 2, \cdots 2000)$ be points in the interior such that no three points from the set

$S = \{A, B, C, D\} \subset \{T_i = 1, 2, \cdots 2000\}$ are collinear.

Prove that at least one triangle with the vertices in S has area less than $\frac{1}{10}$. (AMTI-2000:F)

1.67: A stranger P visited an island, every inhabitant of which is either a 'Knight' who always tells the truth or a 'Knave' who never tells the truth. He met four inhabitants, A, B, C, D. A said: "Exactly one of us is a knave"; B said: "We are all knaves".
Then P asked C: "Is A a knave"? He got an answer (yes or not) from which it was impossible to deduce the truth about A. Is D a knave? (AMTI-1997:F)

1.68: Show that $5^n + 5^k$ where n, k are positive integers can never be the square of an integer.

1.69: Let a, b, c, d be real numbers such that

$a^2 + b^2 + (a-b)^2 = c^2 + d^2 + (c-d)^2$. Prove that

$a^4 + b^4 + (a-b)^4 = c^4 + d^4 + (c-d)^4$ (AMTI-1997:F)

1.70: It is known that 3^{1000} contains 478 digits. Let a be the sum of the digits of 3^{1000}, b the sum of the digits of a and c the sum of the digits of h. Find the value of c.
(AMTI-1997:F)

1.71: Find a number, consisting of digits $1, 2, 3, \cdots 9$ in some order which, when multiplied by 8, still consists of these 9 digits. (AMTI-1997:F)

1.72: The set S consists of 5 integers. If pairs of distinct elements of S are added, the following 10 sums are obtained. 1967, 1972, 1973, 1973, 1974, 1975, 1980, 1983, 1984, 1989, 1991.
Find the elements of S. (AMTI-1997:F)

1.73: Find all integers m, n such that $2mn - 5m + n = 55$ (AMTI-1997:F)

1.74: How many zeroes are there in the number which is made with integers from 1 to 1000 written on paper? (AMTI-2002)

1.75: Find the number of 10-digit numbers, whose sum of the digits is 2. (AMTI-2002)

1.76: Three boys agree to divide a log of marbles as follows: The first boy takes one more than half the marbles. The second boy takes one third of the remaining marbles. The third boy takes the marbles still left out in the bag.

Prove that the original number of marbles (found in the bag in the beginning) should have been two more than a multiple of 6. (AMTI-2002)

1.77: Find the sum of the digits of the number $1000^{20} - 20$ expressed in decimal notation. (AMTI-2002)

1.78: Four digit numbers are formed using the digits 1,2,3,4 (repetitions among the digits allowed).

Find the number of such four digit numbers divisible by 11. (AMTI-2002)

1.79: Given that

$$|x| = \begin{cases} x, & if \quad x \geq 0 \\ -x, & if \quad x < 0 \end{cases}$$

(a) Find the number of real roots of the quadratic equation $x^2 + 2|x| + 1 = 0$. (AMTI-2002)

(b) Find the value of $[\frac{1}{4}] + [\frac{1}{4} + \frac{1}{50}] + [\frac{1}{4} + \frac{2}{50}] + \cdots + [\frac{1}{4} + \frac{40}{50}]$ where the symbol $[x]$ denotes the largest integer less than or equal to x. (AMTI-2002)

1.80: Find the quotient and remainder when $x^{2002} - 2001$ is divided by x^{91}.

1.81: For $x^2 + 2x + 5$ to be a factor of $x^4 + px^2 + q$, find the values of p and q. (AMTI-2002)

1.82: A and B had a pack of playing cards. A gave B some cards, took some himself and then laid the rest of the pack on the floor. B had more cards than A. If A gave a certain number of his cards to B, then B would have four times as many cards as A. But, instead, if B gave the same number to A, then B would be left with three times as many cards as A. How many cards did B give to A in the beginning? (AMTI-2002)

1.83: In a school of student strength 500, two-thirds of the students, who do not wear spectacles do not bring lunch, Three-quarters of the students, who do not bring lunch, do not wear spectacles. Altogether 60 spectacled students bring lunch too. Find the number of students who do not wear spectacles and do not bring lunch. (AMTI-2002)

1.84: Men, women and children, numbering 100 in all were distributed Rs.1000. Each man got Rs.50, Each woman Rs.10 and each child got Rs.0.50. How many children were there? (AMTI-2002)

1.85: P and Q are school students. If one subtracts Q 's age from the square of P 's age, the result is 158 but if P 's age is subtracted from the square of Q 's age, the result is 108. What is the age of P? (AMTI-2002)

1.86: $P(x)$ is a polynomial satisfying $P(x+\frac{3}{2}) = P(x)$ for all real values of x. If $P(5) = 2006$, find $P(8)$. (AMTI-2002)

1.87: If a, b, c are real numbers such that $a + (\frac{1}{b}) = \frac{7}{3}$; $b + (\frac{1}{c}) = 4$; $c + (\frac{1}{a}) = 1$, find abc.

1.88: If $\{(3(230+x)^2)\} = 492a04$, find the value of a. (AMTI-2001)

1.89: Find the number of pythagorean triples (a, b, c) where $a^2 + b^2 = c^2$ and each integer a, b, c being odd. (AMTI-2001)

1.90: In the year 2002, Great Britain hosted the International mathematical olympiad. Let I, M, O be distinct positive integers such that the product $I.M.O = 2002$. Find the largest possible value of the sum $I + M + O$. (AMTI-2001)

1.91: At the end of the year 1994, Ram was half as old as his grandfather while the sum of the years in which they were born is 3844. Find the age of Ram at the end of 2001. (AMTI-2001)

1.92: If a, b, c are real numbers such that
$(2a-3)^2+(4b-5)^2+(6c-7)^2=0$, then, prove that $abc < a+b+c$. (AMTI-2001)

1.93: If the number $N = 30a0b03$ is divisible by 13, then prove that a can take the values 3,2 and 0 but not 1. (AMTI-2001)

1.94: Consider the series $1-4+7-10+13-\cdots$ where the $(2n)^{th}$ term is $-2(3n-1)$ and the $(2n-1)^{th}$ term is $(6n+1)$. Find the sum to first 2001 terms. (AMTI-2001)

1.95: There are 20 people around a table. Each of them shakes hands with the people to his or her immediate right and left. Find the number of hand shakes that take place. (AMTI-2001)

1.96: Between 5p.m and 6p.m, I looked at my watch. Mistaking the hour-hand for the minute-hand, I took the time to be 57 minutes earlier than the correct time. Find the correct time actually? (AMTI-2001)

1.97: Find the number of real roots of the equation

$$2X^{2001}+3X^{2000}+2X^{1999}+3X^{1998}+\cdots+2X+3=0$$

(AMTI-2001)

1.98: In solving a quadratic equation, one student copied the equation wrongly making an error only in the constant term and obtained 8,2 as roots. Another student also copied the equation wrongly making an error only in the coefficient of the first degree term and obtained $-9, -1$ as roots. find the correct equation. (AMTI-2001)

1.99: For a class, copies of 9 mathematics books and 16 science books cost Rs.220/-. Each book costs an integral number of rupees - what is the cost of each mathematics book? (AMTI-2001)

2.00: A natural number is good if it can be expressed both as a sum of two consecutive natural numbers and as a sum of three consecutive natural numbers. Show that

(i) 2001 is good but 3001 is not

(ii) The product of two good numbers is good

(iii) If the product of two numbers is good, then, at least one of them is good. (AMTI-2001:F)

2.01: Find all real numbers a and b such that $x^2 + ax + b^2 = 0$ have at least one common root.

(AMTI-2001:F)

2.02: A school has 281 boys and girls from seven countries. Suppose among any six students, there are at least two who have the same age. Prove that there are five boys from the same country having the same age or there are five girls from the same country having the same age.

(AMTI-2001:F)

2.03: You are given 10 segments such that, each segment is longer than one cm but shorter than 55 cms. Prove that you can select three sides of a triangle among the given segments. (AMTI-2001:F)

2.04: Find the number of positive integer x which satisfy the condition $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right]$. (RMO-2001)

2.05: Prove that the product of the first 1000 positive even integers differs from the product of the first 1000 positive odd integers by a multiple of 2001. (RMO-2001)

2.06: Determine the least positive value taken by the expression $a^3 + b^3 + c^3 - 3abc$ as abc vary over all positive integers. Find also all triples (a, b, c) for which this least value is attained. (INMO:2002)

2.07: Do there exist three distinct positive numbers a, b, c such that a, b, c, $b+c-a$, $c+a-b$, $a+b-c$ and $a+b+c$ form a seven term arithmetic progression in some order? (INMO:2002)

2.08: Show that there are no integers a, b, c for which $a^2 + b^2 - 8c = 6$. (AMTI-2004:F)

2.09: The polynomial $ax^3 + bx^2 + cx + d$ has integral coefficients a, b, c, d. If ad is odd and bc is even, show that at least one root of the polynomial is irrational. (AMTI-2004:F)

2.10: Let f be a function from N to R satisfying
(a) $f(1) = 1$
(b) $f(1) + 2f(2) + 3f(3) + \cdots + nf(n) = n(n+1)f(n)$.
Find $f(2004)$. (AMTI-2004:F)

2.11: Let $a_1, a_2, a_3 \cdots a_m$ be a sequence of real numbers. The sum of K - successive terms is called a $K-$ sum. For example, $a_r + a_{r+1} + a_{r+2} + \cdots + a_{r+k-1}$ is a K - sum. In a finite sequence of real numbers, every 7-sum is negative and every 11-sum is positive. Find the largest number of terms in such a sequence. Try to construct such a sequence also. (AMTI-2004:F)

2.12: In the given multiplication, a and b are natural numbers. Find $a+b$. (AMTI-2003:F)

```
      3a
  x   b2
     -----
      70
     140
   -------
    1470
```

2.13: Rahim wants to arrange a party of a certain number of people such that two of participants (whose dates of birth he knows) of the party will have birthdays in the same month. Find the maximum number of people to be invited for the party. (AMTI-2003)

2.14: If $x = 9+4\sqrt{5}$ and $xy = 1$, prove that $\frac{1}{x^2}+\frac{1}{y^2} = 322$ (AMTI-2003)

2.15: In a kilometer race, A beats B by 1 minute and beats C by 375 meters. If B beats C by 30 seconds, find the time taken by C to run 1km. (AMTI-2003)

2.16: If $a+b+c=0$, find $\frac{b^2+c^2+a^2}{b^2-ca}$ (AMTI-2003)

2.17: At the end of the year 2002, Ram was half old as his grandpa. The sum of the years in which they were born is 3854. What is the age of Ram at the end of the year 2003? (AMTI-2003)

2.18:1 multiplied a natural number by 18 and another natural number by 21 and then added the products. Show that 2004 could be the sum of these products but 2005 or 2006 cannot be natural numbers. (AMTI-2003)

2.19: One hundred and twenty students take an examination which is marked out of a total 100 (with no

fractional marks). No three students are awarded the same mark.
Find the smallest possible number of pairs of students who are awarded the same mark? (AMTI-2003)

2.20: 27 metal balls, each of radius r are melted together to form one big sphere of radius R. Find the ratio of the surface area of the big sphere to that of the ball. (AMTI-2003)

2.21: The product of the ages of three sisters is 36. The sum of their ages is a prime number. The youngest sister likes ice cream. Find the product of the ages of the two elder sisters. (AMTI-2003)

2.22: Given that $x_1, x_2, x_3, \cdots x_{15}, x_{16}$ are positive real numbers such that $\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \cdots = \frac{x_{15}}{x_{16}}$;
if $x_1 + x_2 + x_3 + x_4 = 20$, $x_5 + x_6 + x_7 + x_8 = 320$ find $x_{13} + x_{14} + x_{15} + x_{16}$. (AMTI-2003)

2.23: Find the number of values of K for which the systems of equations $x + y = 2, kx + y = 4, x + ky = 5$ has at least one solution.

2.24: If $S_n = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots$
(up to n terms), find $S_{2002} - S_{2003} + S_{2004}$. (AMTI-2003)

2.25: The sum of all five digit numbers that can be formed using the digits 1,2,3,4 and 5 (repetition of digits not allowed) is x. Find x. (AMTI-2003)

2.26: If $y \neq 0$, find the number pairs (x, y) such that $x + y + \frac{x}{y} = \frac{1}{2}$ and $(x + y)\frac{x}{y} = \frac{-1}{2}$ (AMTI-2003)

2.27: The statement "There are exactly four integer values of n for which $\frac{2n+y}{n-2}$ is itself an integer" is true for certain values of y only. For how many values of y in the range $1 \leq y \leq 20$ is the statement true? (AMTI-2003)

2.28: Let n ne a positive integer with all digits equal to 5 such that n is divisible by 2003. Find the last six digits of $\frac{n}{2003}$.

2.29: Find all pairs (x, y) where (x, y) are integers such that $x^3 + 11^3 = y^3$. (AMTI-2003:F)

2.30: If the quadratic ax^2+bx+c takes rational values for more than two rational vales of x, then, show that a, b, c are all rational numbers. (AMTI-2003:F)

2.31: Let A consist of 16 elements of the set $\{1, 2, 3, \cdots 106\}$ so that two elements of A differ by 6,9,12,15,18 or 21. Prove that two elements of A should differ by 3. (AMTI-2003:F)

2.32: (22,48),(61,76),(29,34) are some pairs of distinct two digit numbers whose product ends with 6. How many such pairs are possible? (AMTI-2005)

2.33: The digits 1,2,3,4 are used to generate 256 different 4 digit numbers. Find the sum of all these 256 numbers. (AMTI-2005)

2.34: Numbers with two digits or more in which the digits reading from left to right occur in strictly increasing order, are called as 'sorted Numbers'. For example 125,14,239 are sorted numbers while 255,74,198 are not. Suppose that a complete list of sorted numbers is prepared and written in

increasing order, find the 100th number on this list.

(AMTI-2005)

2.35: Given

$$\begin{aligned} a + a^2 + a^3 + \cdots &= \frac{5}{6} \\ a^2 + a^3 + a^4 + \cdots &= \frac{25}{66} \\ b + b^2 + b^3 + \cdots &= \frac{6}{5} \\ b^2 + b^3 + b^4 + \cdots &= \frac{36}{55} \end{aligned}$$

Prove $11(a^2 - b^2) + 1 = 0$. (AMTI-2005)

2.36: Given the equation of the circle $x^2 + y^2 = 100$ find the number of points (a, b) lying on the circle, where 'a' and 'b' are integers.

(Note: Such points (a, b) with integral co-ordinates satisfying the given condition, are called *lattice points*).

(AMTI-2005)

2.37: If the points (0,0), (2,0),(3,1),(1,2),(3,3),(4,3) and (2,4) at most how many can be on a circle? Find the equation of this circle also. (AMTI-2005)

2.38: The roots of the equation $x^5 - 40x^4 + Px^3 + Qx^2 + Rx + S = 0$ are in geometric progression. The sum of their reciprocals is 10. Find the value of $|S|$. (AMTI-2005)

2.39: Prove that the equation $x^2 + y^2 + z^2 = 3xyz$ has infinitely many solutions in positive integers. (AMTI-2005:F)

2.40: Let $a_1, a_2, a_3 \cdots a_n$ be rearrangement of numbers $1, 2, 3 \cdots n$. If n is odd, examine whether $(a_1 - 1)(a_2 - 2) \cdots (a_n - n)$ is even. (AMTI-2005)

2.41: Suppose for integers (x, y), we have $2x + 3y$ is divisible by 17. Prove that $9x + 5y$ is also divisible by 17. (AMTI:1995:F)

2.42: Show that there are no solutions in integers such that $\frac{14x+5}{9}$ and $\frac{17x-5}{12}$ are both integers. (AMTI:1995:F)

2.43: Between the digits 4 and 9, several fours and after them the same number of eights were inserted. For example, between 4 and 9, we may insert three fours followed by three eights 444888 to get 44448889. Prove that the resulting number is always a perfect square. (AMTI:2005:F)

2.44: Prove that $abc(a^3 - b^3)(b^3 - c^3)(c^3 - a^3)$ is divisible by 7 where a, b, c are integers.

2.45: If $a \neq 0, b \neq 0, c \neq 0$ and if

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} + \frac{1}{a+x} &= 0 \\ \frac{1}{a} + \frac{1}{c} + \frac{1}{a+y} &= 0 \\ \frac{1}{a} + \frac{1}{x} + \frac{1}{y} &= 0. \end{aligned}$$

Prove that $a + b + c = 0$. (AMTI:2005:F)

2.46: If $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}$,

prove that $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = \frac{1}{a^3 + b^3 + c^3}$. (AMTI:2006:F)

2.47: Let $1 < a_1 < a_2 \cdots < a_{51} < a_{142}$. Prove that among the 50 consecutive differences $(a_i - a_{i-1}), i = 2, 3, 4 \cdots 51$, some value must occur at least 12 times. (AMTI:2006:F)

2.48: Prove that in any perfect square, the three digits immediately to the left of the unit digit cannot be 101.

(For example $\cdots 101x$ cannot be a perfect square).

(AMTI-2006:F)

2.49: $S = [n+\frac{20}{100}]+[n+\frac{21}{100}]+[n+\frac{22}{100}]+\cdots[n+\frac{99}{100}] = 1606$, where '$n$' is a real number and $[x]$ denotes the greatest integer not exceeding x. Find $[100n]$. (AMTI:2006:F)

2.50: Find all integer solutions (a, c) of

$$a^4 + 6a^3 + 11a^2 + 6a + 1 = \frac{q(a^2 - 1)(c^2 - 1)}{a^2 + c^2}$$

where q is the product of arbitrary non-negative powers of alternate primes i.e $q = 2^{b_1} \cdot 5^{b_2} \cdot 11^{b_3} \cdots$, where $b_i \geq 0$.

(AMTI:2006:F)

⋆⋆⋆⋆⋆

UNIT 2: GEOMETRY

(including construction geometry and trigonometry)

3.01: The length of one of the legs of a right angled triangle exceeds the length of the other leg by 10 cm but is smaller than that of the hypotenuse by 10 cm. Find the length of the hypotenuse. (AMTI-1997)

3.02: In $\triangle ABC$, BC and CF are medians. $BE = 9\,\text{cm}$, $CF = 12\,\text{cm}$. If BE is perpendicular to CF, find the area of $\triangle ABC$ in sq.cms. (AMTI-1997)

3.03: Nine lines drawn parallel to the base of a triangle divide the other two sides into 10 equal parts and the area into 10 distinct parts.If the area of the largest of these parts is 1997 sq.cms, find the area of the triangle.

(AMTI-1997:F)

3.04: Let E be the midpoint of median AD of $\triangle ABC$. Let DB be produced to P such that $DB = BP$ and let DC be produced to Q such that $DC = CQ$. Let EP cut AB at L and EQ cut AC at M. Find the ratio of the area of the pentagon $BLEMC$ to that of $\triangle ABC$.

(AMTI-1997:F)

3.05: Triangle ABC is isosceles with base AC. Points P and Q are respectively in CB and AB such that $AC = AP = PQ = QB$. Show that $m\angle B = 25\frac{5}{7}^\circ$.

(AMTI-1998)

3.06: In $\triangle ABC$, $AB = AC$ and $\angle\ BAC = 40^\circ$. If O is such that $\angle OBC = \angle OCA$, prove that the measure of $\angle BOC$ is 110°. (AMTI-1998)

3.07: Find the ratio of the area of the equilateral triangle inscribed in a circle to that of a regular hexagon inscribed in the same circle. (AMTI-98)

3.08: Six straight lines are drawn in a plane with no two parallel and no three concurrent. Find the number of regions into which they divide the plane. (AMTI-98)

3.09: The measures of length of the sides of a triangle are integers and that of its area is also an integer. One side is 21 and the perimeter is 48. Find the measure of the shortest side. (AMTI-98 I) (RMO)

3.10: ABC is an acute angled triangle and from a point D on BC, perpendiculars DE, DF are drawn to AC, AB respectively. Show that EF has minimum length where D coincides with the foot of the perpendicular from A on BC. (AMTI-98 F)

3.11: ABC is an isosceles triangle with $\angle B = \angle C = 78°$. D and E are points on AB, AC respectively such that $\angle BCD = 74°$ and $angle CBE = 51°$. Find $\angle BED$.

(AMTI-98:F)

3.12: Let H be the orthocentre of the triangle ABC and X, Y feet of the perpendiculars from H to the internal and external bisectors of $\angle A$. Let M be the midpoint of BC. Prove that M, X, Y are collinear. (AMTI-98:F)

3.13: In $\triangle ABC$, $BC = 20$, median $BE = 18$ and median $CF = 24$ (E, F are midpoints of AC, AB respectively). Find the area of $\triangle ABC$. (AMTI-98:F)

3.14: Let $ABCD$ be a convex quadrilateral in which $\angle BAC = 50°$, $\angle CAD = 60°$, $\angle CBD = 30°$, $\angle BDC = 25°$; If E is the point of intersection of AC and BD, find $\angle AEB$. (RMO 1998)

3.15: Let ABC be an acute angled triangle in which D, E and F are points on BC, CA, AB respectively such that $AD \perp BC$, $AE = EC$ and CF bisects $\angle C$ internally. Suppose CF meets AD and DE in M and N respectively.
If $FM = 2, MN = 1, NC = 3$, find the perimeter of the triangle ABC. (INMO-99)

3.16: Let Γ and Γ' be two concentric circles. Let $ABC, A'B'C'$ be any two equilateral triangles inscribed in Γ and Γ' respectively. If P and P' are any two points on Γ and Γ' respectively, prove that

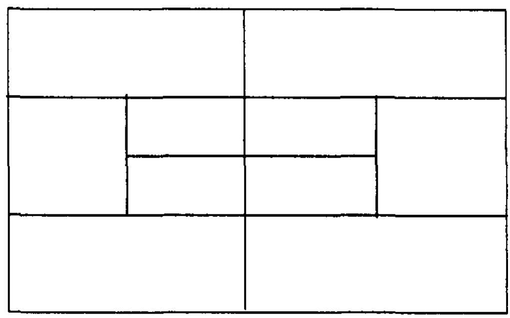
$$P'A^2 + P'B^2 + P'C^2 = A'P^2 + B'P^2 + C'P^2. \quad \text{(INMO-99)}$$

3.17: The line parallel to the parallel sides of a trapezium passing through the midpoints of the slant sides divides the trapezium in the ratio 5:2. Find the ratio of the parallel sides. (AMTI-99)

3.18: Three circles touch each other externally and all the three touch a line. If two of them are equal and the third has radius 4 cm, find the radius of the equal circles.
(AMTI-99)

3.19: In a given polygon, each interior angle is $7\frac{1}{2}$ times the exterior angle at the vertex. Show that the polygon must be 17 sided. (AMTI-99)

3.20: How many rectangles can be realized from the following figure of a tennis court? (AMTI-99:I)



3.21: Using only ruler and compasses, show how to construct two circles to intersect orthogonally (i.e the tangents to the circles at a point of their intersection are perpendicular.) (AMTI-99:F)

3.22: Points M and N lie inside an equilateral triangle ABC such that $\angle MAB = \angle MBA = 40^\circ$; $\angle NAB = 90^\circ$; $\angle NBA = 30^\circ$. Prove that MN is parallel to BC. (AMTI-99:F)

3.23: $ABCD$ is a square with length of a side 1 cm. An octagon is formed by lines joining the vertices of the square to the midpoints of opposite sides. Find the area of the octagon. (AMTI-99:F)

3.24: Prove that the irradius of a right angled triangle with integer sides is also an integer. (RMO-1999)

3.25: Let $ABCD$ be a square and M, N points on sides AB, BC respectively such that $\angle MDN = 45^\circ$. If R is the midpoint of MN, prove that $RP = RQ$ where P, Q are the points of intersection of AC with the lines MD, ND. (AMTI-1999)

3.26: l_1, l_2, l_3 are any three distinct non-concurrent lines in a plane, with no two of them parallel. Find the number of circles for which all of l_1, l_2 and l_3 are tangents.

(AMTI-2000)

3.27: A rectangle contains three circles, all tangent to the rectangle and also to one another. If the height of the rectangle is 4, find the width of the rectangle.

(AMTI-2000)

3.28: Let $ABCD$ be a rectangle with $BC = 3AB$. Show that, if P, Q are points on BC with $BP = PQ = QC$, then, $\angle DBC + \angle DPC = \angle DQC$ (AMTI-2000)

3.29: Two fixed circles touch each other at A. If a variable line through A cuts the circle at P, Q, Prove that $\frac{AP}{AQ}$ is constant. (AMTI-2000)

3.30: The point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180°$. Prove that $\angle OBC = \angle ODC$ (AMTI-2000)

3.31: A ball was floating in a lake in the Himalayas, when the lake froze. The ball was removed (without breaking the ice) having a hole 24 cm across at the top and 8 cm deep. Find the radius of the ball. (AMTI-2001)

3.32: Chords AB and PQ meet at K and are perpendicular to one another. If $AK = 4, KB = 6$ and $PK = 2$, find the area of the circle. (AMTI-2001)

3.33: Points M and N are the midpoints of the sides PA and PB of $\triangle PAB$. As P moves along a line that is parallel to side AB, consider the following.

(i) The length MN

(ii) The area of $\triangle PAB$
(iii) The area of trapezoid $ABNM$
(iv) The perimeter of $\triangle PAB$
which of the above will change? Why? (AMTI-2001)

3.34: In quadrilateral $ABCD$, diagonals AC and BD meet at O. If $\triangle AOB$, $\triangle DOC$ and $\triangle BOC$ have areas 3,10 and 2 respectively, find the area of $\triangle AOD$.

(AMTI-2001)

3.35: Medians BE and CF of $\triangle ABC$ are perpendicular. If $BE = 12$ and $CF = 18$, find the area of $\triangle ABC$. (AMTI-2001)

3.36: Given the rhombus $ABCD$ with $\angle A = 60°$. The points F, H and G are marked on the segments AD, DC and the diagonal AC so that $DFGH$ is a parallelogram. Prove that the triangle FBH is equilateral.

(AMTI-2001:F)

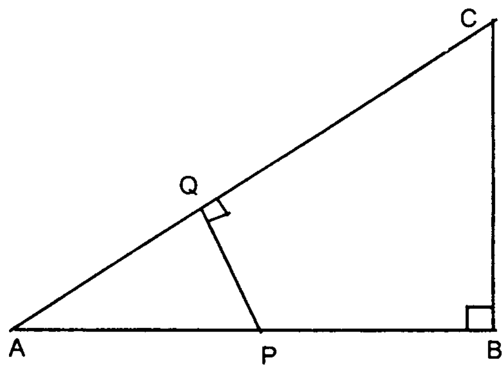
3.37: Suppose the angle formed by the two rays OX and OY is the acute angle α and A is a given point on the ray OX. Consider all circles touching OX at A and intersecting OY at B, C. Prove that the incentres of all triangles ABC lie on the same st.line. (AMTI-2001:F)

3.38: Let BE and CF be the altitudes of an acute triangle ABC, with E in AC and F in AB. Let O be the point of intersection of BE and CF. Take any line KL through O with K on AB and L on AC. Suppose M and N are located on BE and CF respectively such that KM is perpendicular to BE and LN is perpendicular to CF. Prove that FM is parallel to EN.

(RMO 2001)

3.39: In $\triangle ABC$, D is a point on BC such that AD is the internal bisector of $\angle A$. Suppose $\angle B = 2\angle C$ and $CD = AB$. Prove that $\angle A = 72°$. (RMO 2001)

3.40: In the figure, $\angle Q$ and $\angle B$ are right angles. If $AQ = 15$, $BC = 16$, $AP = 17$, find QC. (AMTI-2002:F)



3.41: In a right angled triangle, if the square of the hypotenuse is twice the product of the other two sides, prove that the triangle is isosceles. (AMTI-2002:F)

3.42: If a sector of a circle rotates around the centre of the circle in movements of $75°$, find the number of sector movements needed for the sector to come back to its original position. (AMTI-2002)

3.43: A convex polygon has 44 diagonals. Find the number of sides of this polygon. (AMTI-2002)

3.44: Find the area of the largest square which can be inscribed in a right angled triangle with legs 4 and 8. (AMTI-2004)

3.45: Two circle with centres A and B and radius 2 touch each other externally at C. A third circle with centre C and radius 2 meets the other two at D, E. Find the area $ABDE$. (AMTI-2004)

3.46: In $\triangle ABC$, $\angle A = 90^\circ$ and I is its incentre. The perpendicular distance of I from BC is $\sqrt{8}$. Find the length of AI. (AMTI-2004)

3.47: Prove that, in an isosceles triangle, the centroid, the orthocentre, the incentre and the circumcentre are collinear. (AMTI-2004)

3.48: A circle and a parabola are drawn on a sheet of paper. Find the number of regions they divide the paper into. (AMTI-2004)

3.49: Let ABC be a triangle with $AC > BC$. Let D be the midpoint of the arc AB that contains C, on the circumcircle of $\triangle ABC$. Let E be the foot of the perpendicular from D on AC.
Prove that $AE = EC + CB$. (AMTI-2002:F)

3.50: Given three non-collinear points A, B, C, construct a circle with centre C such that the tangents from A and B to the circle are parallel.

3.51: Given a circle with diameter AB and a point X on the circle different from A and B; let t_a, t_b and t_x be the tangents to the circle at A, B and X respectively. Let Z be the point where the line AX meets t_b and Y the point where the line BX meets t_a. Prove that the three lines YZ, t_x and AB are either concurrent or parallel.

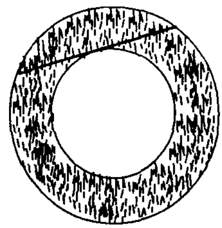
3.52: Points P, Q, R, S divide the sides of a rectangle in the ratio 1:2 with points P, Q, R, S being nearer to A, B, C, D respectively. Find the ratio of the area of

the rectangle to the area of parallelogram $PQRS$?

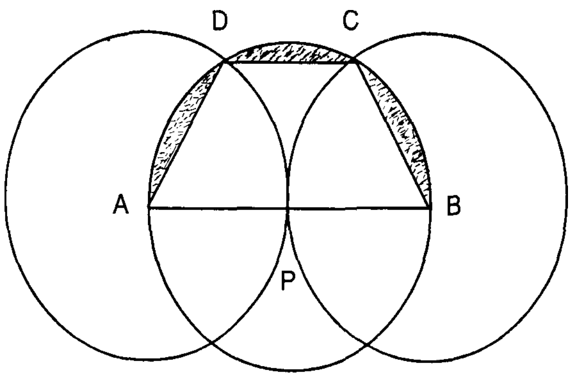
(AMTI-2003)

3.53: The diagram shows two concentric circles. The chord of the larger circle is a tangent to the smaller circle and has length 2p. Find the area of the shaded region?

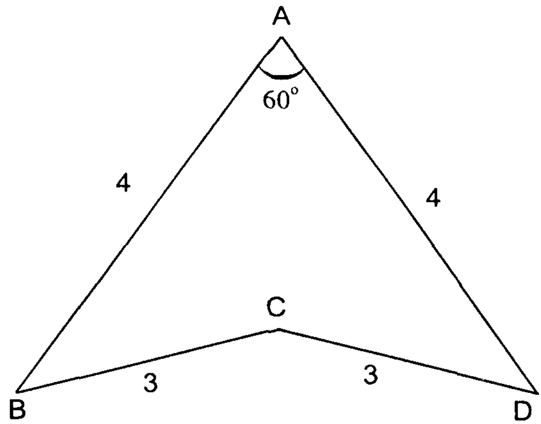
(AMTI-2003)



3.54: AB is a line segment of length 4 cm. P is the midpoint of AB. Circles are drawn with A, P, B as centres and radii $AP = PB$. Find the area of the shaded portion in the figure. (AMTI-2003)



3.55: In the figure, $ABCD$ is a non-convex quadrangle. If $\angle BAD = 60°$, $AB = 4 = AD$ and $BC = 3 = DC$, find AC. (AMTI-2003)



3.56: ABC is a triangle for which BE, CF are medians. If $BE = 15$ cm, $CF = 36$ cm and $BE \perp CF$, find the area of $\triangle ABC$.

3.57: $\triangle ABC$ has a right angle at A. Among all points P on the perimeter of the triangle, find the position of P such that $AP + BP + CP$ is minimized. (AMTI-2003:F)

3.58: Within a given regular pentagon $ABCDE$, draw 10 equal circles, such that, each side of the pentagon touches three of them and each circle touches the two adjacent circles. (AMTI-2003:F)

3.59: Consider a convex quadrilateral $ABCD$ in which K, L, M, N are the midpoints of the sides AB, BC, CD, DA respectively. Suppose

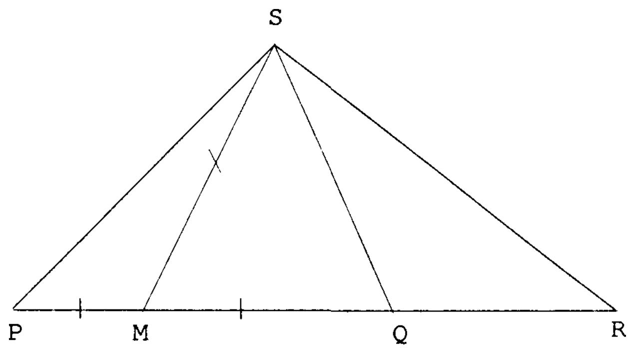
(a) BD bisects KM at Q

(b) $QA = QB = QC = QD$ and

(c) $\dfrac{LK}{LM} = \dfrac{CD}{CB}$.

Prove that $ABCD$ is a square. (INMO 2004)

3.60: In the figure P, M, Q and R are collinear points and $PM = MQ = MS$. Also $SR^2 = PR \cdot QR$. Prove $\angle QSR = \angle MSP$. (AMTI-2005)



3.61: $PQRS$ is a common diameter of three circles. The area of the middle circle is the average of the areas of the other two. If $PQ = 2, RS = 1$, find the length of QR.

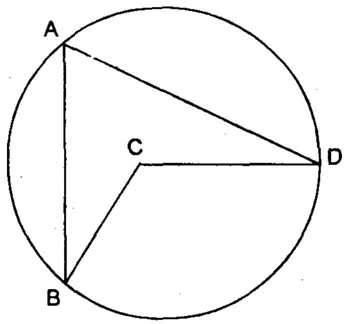
(AMTI-2005)

3.62: The side AB of the parallelogram $ABCD$ is produced both ways to P and Q such that

$PA = AB = BQ$. Given that $AD = 2AB$, find the angle between DQ and CP. (AMTI-2005:SJ)

3.63: A walker's path in a park is as shown in the diagram. One can walk all along the circumference or along the line segments marked in the figure. A child walks along the circumference-path starting at A and after going round returns to A. Her grandfather starts at A and walks

through the route $AB \rightarrow BC \rightarrow CD \rightarrow DA$ and thus returns to A. Who walked more distance. the child or grandpa? (AMTI-2005:SJ)



3.64: A triangle ABC has its circumcentre at O and M is the midpoint of the median through A. If OM is produced to N such that $OM = MN$, prove that N lies on the altitude through A. (AMTI-2005:SJ,RMO)

3.65: Two sides of a triangle are $\sqrt{3}$ cms and $\sqrt{2}$ cms. The medians to these sides are perpendicular to each other. Find the third side. (AMTI-2005:F)

3.66: O is the circumcentre of $\triangle ABC$ and K is the circumcentre of $\triangle AOC$. The lines AB, BC meet the circle $A \cup C$ again at M and N respectively. L is the reflection of K in the line MN. Find the angle between BL and AC. (AMTI-2005:F)

3.67: In a non-equilateral triangle ABC, the sides a, b, c form an Arith an Arithmetic progression. Let I and O denote the in-centre and circumcentre of the triangle respectively.

(i) Prove that IO is perpendicular to BI.

(ii) Suppose BI extended meets AC in K and D, E are the midpoints of BC, BA respectively. Prove that I is the circumcentre of ΔDKE. (INMO-2006)

3.68: Let $ABCD$ be a convex quadrilateral. P, Q, R, S be the midpoints of AB, BC, CD, DA respectively such that the triangles AQR and CSP are equilateral. Prove that $ABCD$ is a rhombus. Determine its angles.

(CRMO-2005)

3.69: In ΔABC, let D be the midpoint of BC. If $\angle ADB = 45°$ and $\angle ACD = 30°$, determine $\angle BAD$.

(CRMO-2005)

3.70: In ΔABC, $\angle A = 75°$, $\angle B = 60°$, $\angle C = 45°$. Also CF and AD are the altitudes from C and A respectively. If H is the orthocentre and O is the circumcentre, prove that, O is the incentre of ΔCHD. (AMTI-2006:F)

3.71: For a convex hexagon $ABCDEF$, consider the following six statements.

(a_1) AB is parallel to DE $\quad$ $(a_2) AE = BD$

(b_1) BC is parallel to EF $\quad$ $(b_2) BF = CE$

(c_1) CD is parallel to FA $\quad$ $(c_2) CA = DF$

(a) Show that, if all the six statements are true, then the hexagon is cyclic.

(b) Prove that, in fact, any five of the six statements also imply that the hexagon is cyclic. (INRO-2002)

3.72: Assume that ΔABC is isosceles with $\angle ABC = \angle ACB = 78°$. Let D and E be points on

sides AB and AC respectively so that $\angle BCD = 24°$ and $\angle CBE = 51°$. Find $\angle BED$ and justify. (AMTI-1999:F:I)

3.73: ABC and DAC are two isosceles triangles with $\angle BAC = 20°$ and $\angle ADC = 100°$.
Show that $AB = BC + CD$. (AMTI-1999:F:I)

3.74: Let $ABCD$ be a quadrilateral inscribed in a circle. Let M be the point of intersection of the diagonals AC and BD and let E, F, G and H be the feet of the perpendiculars from M on the sides AB, BC, CD, DA respectively. Find the centre of the circle that can be inscribed in the quadrilateral $EFGH$.
(i.e touching all its sides). (AMTI-1999:F:I)

3.75: Let $ABCD$ be a cyclic quadrilateral. Points C_1, A_1 are marked on the rays BA and DC respectively so that $DA = DA_1$ and $BC = BC_1$. Prove that the diagonal BD intersects the segment A_1C_1 at its midpoint.
(AMTI-2001:F:I)

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UNIT 3: NUMBER SYSTEM

4.01: Find the smallest positive number from the numbers below $10-3\sqrt{11}, 3\sqrt{11}-10$, $18-5\sqrt{13}$, $51-10\sqrt{26}$, $10\sqrt{26}-51$ (AMTI-1998)

4.02: Evaluate: $\frac{2\sqrt{6}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$ (AMTI-1998)

4.03: Find the largest integer n for which $n^{200} < 5^{300}$ (AMTI-1998)

4.04: Show that the number represented by

$\sqrt{3+2\sqrt{2}}-\sqrt{3-2\sqrt{2}}$ is an integer. (AMTI-1998)

4.05: The sides of a right angled triangle are $a, a+d$ and $a+2d$ with a and d both positive. Find the ratio of a to d. (AMTI-1998)

4.06: Prove that $\frac{\sqrt{\sqrt{5}+2}+\sqrt{\sqrt{5}-2}}{\sqrt{5}+1}-\sqrt{3-2\sqrt{2}}$ is a positive integer. (AMTI-1998)

4.07: Find the smallest positive integer x for which $1260x = N^3$ where N is an integer (AMTI-1998)

4.08: How many three digit numbers are there, the sum of whose digits is odd? (AMTI-1998)

4.09: $a679b$ is a 5-digit number in decimal system(base ten) which is divisible by 72. Find a and b

(AMTI-1998:F)

4.10: Find the last two digits of $7^{7^{7^{7}}}-7^{7^{7}}+7^{7}-7$

(AMTI-1998:F)

4.11: Let n be a positive integer and $p_1, p_2, \cdots p_n$ be n prime numbers, all larger than 5, such that, $p_1^2+p_2^2+\cdots+p_n^2$ is divisible by 6. Prove that 6 divides n.

4.12: Let M be a 5-digit number and N be the number obtained from M by writing the digits of M in the reverse order. Prove that, at least one digit in the decimal representation of the sum $M+N$ is even. (AMTI-1997:F)

4.13: If $x^9 = 9^{9^9}$, what is x? (AMTI-1999)

4.14: If $1\times2\times3\times n\cdots39\times40$ is written in the system with base 13, find the number of zeros with which the resulting number ends. (AMTI-1998)

4.15: Find the number of prime numbers p such that $1999!+1 < p < 1999!+1999$. (AMTI-1998)

4.16: Which of the following numbers is different from the remaining?

$(a)\dfrac{1996}{1997}$ $(b)\dfrac{996}{997}$ $(c)\dfrac{19979966}{1998997}$ $(d)\dfrac{19971996}{19981997}$ $(e)\dfrac{996996}{997997}$.

(AMTI-99)

4.17: Find the smallest natural number n such that $n!$ is divisible by 1000. (AMTI-99)

4.18: Find the number of divisors of the sum of all two-digit numbers which leave remainder 1 when divided by 5. (AMTI-99)

4.19: Find the sum of the digits of the number $1000^{20}-20$ expressed in decimal notation. (AMTI-99)

4.20: Find the last two digits of 6^9. (AMTI-99)

4.21: Show that there exists no integer n such that the sum of the digits of n^2 is 2000. (AMTI-99:F)

4.22: Show that $4^{1999} + 7^{1999} - 2$ is divisible by 9 (AMTI-99:F)

4.23: Let n be a positive integer greater than 5. Show that, atmost eight numbers of the set $\{n+1, n+2 \cdots n+30\}$ can be primes. (AMTI-99:F)

4.24: Starting with the four-digit number N in base 10, we subtract a 3-digit number formed by dropping of the last digit (on the right) of N, and then, we add the 2-digit number formed by dropping the last 2 digits of N and add the one-digit number formed by dropping the last three digits of N. What is N if the above computations yield 1999? (AMTI-99:F)

4.25: Find all integers n such that $\frac{n^3-1}{5}$ is a prime number. (AMTI-99:F)

4.26: Find the number of positive integers which divide 10^{999} but not 10^{998}. (RMO 99)

4.27: The natural numbers $1, 2, 3 \cdots$ up to 100 are written in that natural order to form the single number $1234 \cdots 979899100$. The number of times the digit 1 occurs is x. Find x. (AMTI-2000)

4.28: In the sequence $1, 23, 45, 67, 89, 1011, 1213 \cdots 9899, 100101, 102103 \cdots$ (from the second term onwards, two consecutive natural numbers written together in the natural order, form the terms). Find the number of terms having four digits. (AMTI-2000)

4.29: If $n! = 1 \times 2 \times 3 \times \cdots \times n$ has four zeroes at the end and $(n+1)! = n! \times (n+1)$ has six zeroes at the end, find the value of n. (AMTI-2000)

4.30: Given that $x^2 - y^2 = 1$ where x, y are integers, find the solution $(x.y)$ for the given equation. (AMTI-2000)

4.31: $x_1, x_2, x_3 \cdots x_{10}$ are integers none of which are divisible by 3. Find the remainder when $x_1^2 + x_2^2 + x_3^2 + \cdots + x_{10}^2$ is divisible by 3. (AMTI-2000)

4.32: Given natural numbers a, b, c such that $a^3 - b^3 - c^3 = 3abc$ and $a^2 = 2(b+c)$, find the values of a, b and c; (a, b, c need not be all different). (AMTI-2000:F)

4.33: We multiply the consecutive even positive integers until the product $2 \cdot 4 \cdot 6 \cdot 8 \cdots$ becomes divisible by 2001. Find the largest even integer used for this purpose.

4.34: The unit of digit of a square number is 6. Find the ten's digit of that number. (AMTI-2002:F)

4.35: We define "funny numbers" as follows: Every single digit prime is funny. A prime number with 2 or more digits is funny, if the two numbers obtained by deleting either its leading digit or its unit's digit are both funny. Find all funny numbers. (AMTI-2002:F)

4.36: The difference between two prime numbers is 100. If we concatenate (justapose one number into another)them in one order, we obtain another prime number.
Find the two primes and the concatenated prime.
(AMTI-2002:F)

4.37: When a number is removed from a finite collection of consecutive natural numbers, the average of the remaining is 50.55. Find all possible collection of such natural numbers and the number removed. (AMTI-2002:F)

4.38: A sports meet was organized for four days. If on each day, half of the existing medals and one more medal was awarded, find the number of medals awarded on each day. (AMTI-2002:F)

4.39: If the numbers 2^{2001} and 5^{2001} are written one after another (in decimal notation), then find the total number of digits written altogether. (AMTI-2001)

4.40: All odd numbers from 1 to 99 (both inclusive) are multiplied together. Find the two right most digits of the product. (AMTI-2001)

4.41: Let A be the set of all 7-digit numbers with different digits $1, 2, 3, 4, 5, 6$ and 7. Prove that there are no two numbers in A such that one of them is divisible by the other. (AMTI-2001:F)

4.42: Find all primes p and q such that $p^2 + 7pq + q^2$ is the square of an integer. (RMO 2001)

4.43: Consider an $n \times n$ array of numbers

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{array}$$

Suppose each now consists of the n numbers $1, 2, 3, \cdots n$ in some order and $a_{iJ} = a_{Ji}$ for
$i = 1, 2, \cdots n$ and $J = 1, 2, \cdots n$. If n is odd, prove that the numbers $a_{11}, a_{22}, a_{33} \cdots a_{nn}$ are $1, 2, 3 \cdots n$ in some order. (RMO-2001)

4.44: Given that $2^n(2^{n+1} - 1)$ and $2^{n+1} - 1$ is a prime number, show that
(a) sum of the divisors of N is $2N$.
(b) Sum of the reciprocals of the divisors of N is 2.
(AMTI-2004:F)

4.45: Consider a permutation $p_1p_2p_3p_4p_5p_6$ of the six numbers 1,2,3,4,5,6 which can be transformed to 123456 by transposing two numbers exactly four times. By a transposition we mean an interchange of two places - for example, 123456 to 321456 (positions 1 and 3 are interchanged). Find the number of such permutations.
(AMTI-2004:F)

4.46: Given the sequence $a, ab, aab, aabb, aaabbb, \cdots$ upto 2004 terms. Find the total number of times a's and b's are used from 1 to 2004 terms. (AMTI-2004)

4.47: Find the number of two digit numbers divisible by the product of the digits. (AMTI-2004)

4.48: In this addition, each letter represents a different digit. Which is the missing digit?

$$\begin{array}{rccccc} & & A & B & C & D \\ + & & & B & C & D \\ \hline & G & H & I & J & K \end{array}$$

(AMTI-2004)

4.49: Five children, each owned a different number of rupees. The ratio of any one's fortune to the fortune of every child poorer than himself was an integer. The combined fortune of the children was 847 rupees. Find the least number of rupees that a child had. Suggest a possible solution for the fortune owned by every one.

(AMTI-2004:F)

4.50: A number with 8 digits is a multiple of 73 and also a multiple of 137. Find the second digit from the left?

(AMTI-2004:F)

4.51: Let m be the least positive integer such that 1260m is the cube of a natural number.

Show that $1000 < m < 10000$. (AMTI-2004:F)

4.52: If $(43)_x$ in base x number system is equal to $(34)_y$ in base y number system, find the possible value for $x+y$.

(AMTI-2004.F)

4.53: In each of the following 2003 fractions, the sum of the numerator and denominator equals 2004.

$$\left[\frac{1}{2003}, \frac{2}{2002}, \frac{3}{2001}, \frac{4}{2000}, \cdots \frac{2003}{1}\right].$$

Find the number of fractions less than 1 which are irreducible. (AMTI-2004)

4.54: For how many integers n is $\sqrt{9-(x+2)^2}a$ real number? (AMTI-2004)

4.55: Let $[n]$ denote the greatest integer less than or equal to x. Find the value of $[\sqrt{1}]+[\sqrt{2}]+[\sqrt{3}]+\cdots+[\sqrt{2004}]$.

(AMTI-2004)

4.56: How many solutions are there for (a, b) if $7ab73$ is a five digit number divisible by 99? (AMTI-2004)

4.57: Show that the number $107^{90} - 76^{90}$ is divisible by 61. (AMTI-2004)

4.58: A sequence $a_0, a_1, a_2, a_3, \cdots a_n \cdots$ is defined such that $a_0 = a_1 = 1$ and $a_{n+1} = (a_{n-1} \cdot a_n + 1)$ for $n \geq 1$. Which of the following are true?
(a) $4 \not| \, a_{2004}$ (b) $3 \not| \, a_{2004}$ (c) $5/a_{2004}$ (d) $2 \not| \, a_{2004}$. (AMTI-2004)

4.59: Which of the following is the best approximation to

$$\frac{(2^3-1)(3^3-1)(4^3-1)\cdots(1000^3-1)}{(2^3+1)(3^3+1)(4^3+1)\cdots(1000^3+1)}$$

(a)3/5 (b) 33/50 (c) 333/500 (d)3333/5000. (AMTI-2004)

4.60: Find the 100^{th} root of $(10)^{(10^{10})}$ (AMTI-2003)

4.61: C is the set of all sums of cubes of three consecutive natural numbers. Prove that every element of C is divisible by 3. (AMTI-2003)

4.62: Three people each think of a number which is the product of two different primes. Find the number which could be the product of three numbers which each one of them thought of. (AMTI-2003)

4.63: A number n leaves the same remainder while dividing 5814, 5430, 5958. What is the largest possible value of n? (AMTI-2003)

4.64: Find the last digit in the finite decimal representation of the number $1/5^{2003}$? (AMTI-2003)

4.65: Let n be a positive integer with all digits equal to 5 such that n is divisible by 2003. Find the last six digits of $n/2003$. (AMTI-2003)

4.66: The digits 1,2,3,4 are used to generate 256 different 4-digit numbers. Find the sum of the 256 numbers. (AMIT-2005)

4.67: Find the last two digits of $(2006)^{2005}$. (AMTI-2005)

4.68: "The number of prime numbers less than 100 which can be expressed as the sum of the squares of two natural numbers is 12" Examine this statement. (AMTI-2005)

4.69: Find the number of ordered pair of digits (A, B) such that $A3640548981270644B$ is divisible by 99. (AMTI-2005)

4.70: Find all prime numbers p and q such that $p(p+1)+q(q+1)=n(n+1)$ for some positive integer n. (AMTI-2005)

4.71: If x, y are integers and 17 divides both the expression $x^2-2xy+y^2-5x+7y$ and $x^2-3xy+2y^2+x-y$, then, prove that 17 divides $xy-12x+15y$. (CRMO-2005)

4.72: Determine all triples (a, b, c) of positive integers such that $a \leq b \leq c$ and $a+b+c+d+bc+ca=abc+1$. (CRMO-2005)

4.73: A six digit number is said to be lucky if the sum of its first three digits is equal to the sum of its last three digits. Prove that the sum of all six digit lucky numbers is divisible by 13. (AMTI-1998 I:F)

4.74: Show that in the year 1996, no one could claim on his birthday, his age was the sum of the digits of the year in which he was born. Find the last year prior to 1996 which had the same property. (AMTI-1998 I:F)

4.75: Determine the missing entries in the magic square shown below, so that the sum of the three numbers in each of the three rows, in each of the three columns and along the two major diagonals is the same constant K. Find also the value K. (AMTI-1998 I:F)

		33
31	28	

4.76: Let $E(m)$ denote the number of even digits in m, for example, $E(2) = 1$, $E(19) = 0$, $E(5672) = 2$ etc. Find $E[E(101) \times E(201) \times E(301) \times \cdots \times E(2001)]$

(AMTI-2001)

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UNIT 4: INEQUALITIES

5.01: "The minimum value of $\sqrt{x^2+y^2}$ if $5x+12y=60$ is $\frac{60}{13}$." - Examine the statement for $x, y \in R$ (AMTI-98:I)

5.02: Find the set of values of x satisfying $2 \le |x-1| \le 5$. (AMTI-98:I)

5.03: Find the smallest value of $\frac{4x^2+8x+13}{6(1+x)}$ for $x \ge 0$. (AMTI-98:I)

5.04: If x satisfies $\sqrt[3]{x+9} - \sqrt[3]{x-9} = 3$, prove that $75 < x^2 < 85$ (AMTI-98:I)

5.05: If x, y are positive real numbers such that $x+y=1$, prove that $(1+\frac{1}{x})(1+\frac{1}{y}) \ge 9$ (AMTI-98:I)

5.06: Four bags were to be weighed but the scale could only weigh weights in excess of 100kg. If the bags were weighed in pairs and the weights were found to be 103,105,106,107 and 109, find the weight of the lightest bag. (AMTI-99:I:F)

5.07: If $0 < a < 1$, prove that the value of the expression

$$\left\{\frac{\sqrt{1+a}}{\sqrt{1+a}-\sqrt{1-a}} + \frac{1-a}{\sqrt{1-a^2}-1+a} \times \sqrt{\frac{1}{a^2}-1} - \frac{1}{a}\right\}$$

lies between -2 and +2. (AMTI-99:I:F)

5.08: 7 persons among themselves have Rs. 332 in all. Some three of them possess a total of at least n rupees. Find the greatest integer n with this property. (AMTI-99:I:F)

5.09: The cost of 175 chocolates is more than that of 125 cups of coffee but is less than that of 126 cups of coffee. Is a

rupee enough to buy a cup of coffee and 3 chocolates if each cost in whole paise? Justify your answer. (AMTI-99:F)

5.10: If a, b, c are the sides of a triangle, prove the following inequality:

$$\frac{a}{c+a-b}+\frac{b}{a+b-c}+\frac{c}{b+c-a} \geq 3 \qquad \text{(RMO 99)}$$

5.11: Show that the real number $r = \frac{\sqrt{3}+\sqrt{5}}{\sqrt{3+\sqrt{5}}}$ satisfies the inequality $\sqrt{2} < r < 2$ (AMTI-2000)

5.12: Exactly five interior angles of a convex polygon are obtuse. Find the maximum possible number of sides for such a polygon. (AMTI-2002)

5.13: Prove or disprove the statement: "The measures of three consecutive angles of a quadrilateral are as 2:3:4. Then the fourth angle must satisfy the inequality $90° < \theta < 180°$." (AMTI-2002)

5.14: Let a, b, c be real numbers such that $a+b+c=1$. Prove that $a^2+b^2+c^2 \geq 4(ab+bc+ca)-1$. When does the equality hold? (AMTI-2002:F)

5.15: Men and women devotees totalling 12, men outnumbering women, visited a Swamiji and sought his blessings. The Swamiji distributed 142 flowers. Each man got same number of flowers while each woman got two more than each man. How many flowers did each woman get? (AMTI-2001)

5.16: If, for $xyz \geq 0$, $x+y+z=11$, find the maximum of $xyz+xy+yz+zx$. (AMTI-2001:I)

5.17: A lady takes her triplets and a younger niece to a restaurant on the birthday of the triplets. The restaurant charges Rs.75 for the mother and Rs.5 for each completed year of a child's age. If the total bill is Rs.160 and if her niece is younger than her triplets, find the age of the triplets. (AMTI-2001:I)

5.18: Let a, b, c be positive real numbers such that $a+b+c \geq abc$. Prove that $a^2+b^2+c^2 \geq \sqrt{3}abc$. (AMTI-2001:F)

5.19: If x, y, z are the sides of a triangle, prove that $|x^2(y-z)+y^2(z-x)+z^2(x-y)| < xyz$. (RMO-2001)

5.20: Let x, y be positive real such that $x+y=2$. Prove that $x^3y^3(x^3+y^3) \leq 2$. (INMO 2002)

5.21: The sum of all angles except one of a convex polygon is $2190°$ (where the angles are less than $180°$). Find the possible number of sides of the polygon. (AMTI-2004)

5.22: If the roots of the equation
$x^2-2ax+a^2+a-3=0$ are real and less than 3, prove $a<2$. (AMTI-2004)

5.23: Let a and b be positive numbers. Prove that

$$\sqrt[3]{\tfrac{a}{b}}+\sqrt[3]{\tfrac{b}{a}} \leq \sqrt[3]{2(a+b)(\tfrac{1}{a}+\tfrac{1}{b})}.$$ (AMTI-2003:F)

5.24: Show that
$\log_4 192 < \log_5 500 < \log_6 1080 < \log_3 108.$

(AMTI-2003:F:I)

5.25: Let x, y, z be positive If the harmonic mean of x, y, z be h and that of $1+x, 1+y, 1+z$ is H, prove that $H \geq 1+h$. (AMTI:2003:I:F)

5.26: Let R denote the circum radius of a triangle ABC; a, b, c its sides BC, CA, AB; n_a, n_b, n_c is exradii opposite to A, B, C. If $2R \geq n_a$, prove that

(i) $a > b$ and $a > c$

(ii) $2R > r_b$ and $2R > r_c$. (INMO 2004)

5.27: In a cyclic quadrilateral $ABCD$, $AB = a$, $BC = b$, $CD = c$, $\angle ABC = 120°$ and $\angle ABD = 30°$. Prove that

(i) $c \geq a + b$

(ii) $|\sqrt{c+a} - \sqrt{c+b}| = \sqrt{c-a-b}$. (INMO 2006)

5.28: If a, b, c are three real numbers such that

$$|a-b| \geq |c|,\ |b-c| \geq |a|,\ |c-a| \geq |b|$$

prove that one of a, b, c is the sum of the other two.

(CRMO-2005)

5.29: Let a, b, c be three positive real numbers such that $a+b+c=1$. Let $\lambda = \min\{a^3 + a^2bc, b^3 + ab^2c, c^3 + abc^2\}$. Prove that the roots of the equation $x^2 + x + 4\lambda = 0$ are real. (CRMO:2005)

5.30: Given that a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3abc$. Prove that,

$$\frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} \geq \frac{9}{a+b+c}.$$ (AMTI-2006:F)

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SOLUTIONS

UNIT 1: ALGEBRA

1.01 If x is the distance between the house and the station, it is given that $\frac{x}{10} - \frac{x}{15} = 2$ i.e., $x = 60$.

If d is the speed at which he should drive to reach the station at 4 p.m, then $\frac{60}{10} - \frac{60}{d} = 1$.

i.e., $$\frac{60}{d} = 5 \text{ or } d = 12.$$

$\therefore$ At 12km per hour he should drive.

1.02 Given

$$\frac{1}{\sqrt{n} + \sqrt{n-1}} < \frac{1}{100}, \text{ i.e., } \sqrt{n-1} + \sqrt{n} > 100 \qquad \text{(A)}$$

i.e., $2\sqrt{n} > 100 \Rightarrow \sqrt{n} > 50 \Rightarrow n > 2500$.

$\frac{1}{\sqrt{n} + \sqrt{n-1}}$ decreases as n increases. If n is the smallest, such that $\frac{1}{\sqrt{n} + \sqrt{n-1}} < \frac{1}{100}$, it is given that

$$\frac{1}{\sqrt{(n-1)} + \sqrt{n-2}} \geq \frac{1}{100} \text{ i.e., } \sqrt{n-1} + \sqrt{n-2} \leq 100.$$

or $2\sqrt{n-2} \leq 100$ (B)

or $(n-2) \leq 2500$ i.e., $n \leq 2502$.

Hence $n = 2501$ which is required.

1.03 Let x men and y women work on that day. Then children brought on that day one $\frac{1}{3}(x+y)$ in number. Trees planted are in total number $13x + 10y + \frac{6}{3}(x+y) = 159$.

i.e., $15x + 12y = 159$ i.e., $5x + 4y = 53$. Integer solution pairs of (x, y) are $(1, 12), (5, 7), (9, 2), (5, 7)$ alone is true since exactly $\frac{1}{3}$ rd of the workers having a child each. Thus women workers are 7 in number.

1.04
$$\binom{-\frac{1}{2}}{100} = \frac{(-\frac{1}{2})(\frac{-3}{2})(-\frac{5}{2})\dots(-\frac{155}{2})}{1.2.3\dots 100}$$

$$\binom{\frac{1}{2}}{100} = \frac{(\frac{1}{2})(-\frac{3}{2})(-\frac{3}{2})\dots(-\frac{151}{2})}{1.2.3\dots 100}$$

$\therefore$ The defined quotient $= -\frac{199}{2}/\frac{1}{2} = -199$.

1.05 If A, B, C are the speeds of A, B, C respectively.

$$\frac{d}{A} = \frac{d-30}{B}, \qquad \text{(i)}$$
$$\frac{d}{B} = \frac{d-20}{C}, \qquad \text{(ii)}$$
$$\frac{d}{A} = \frac{d-48}{C} \qquad \text{(iii)}$$

From the equations (i) and (ii),

$$\frac{d-30}{B} = \frac{d-48}{C}\text{ ; using (ii)}$$

$$\frac{d-30}{\frac{dC}{d-20}} = \frac{d-48}{C} \Rightarrow (d-30)(d-20) = d(d-48)$$

$$\therefore \quad d = 300 \text{ metres}$$

1.06 In one stretch there will be $9 + (90 \times 2) + (900 \times 3)$ digits. Giving upto 2 digits numbers, there one 189 numbers.

$$1983 - 189 = 1794.$$

$$\therefore \quad \frac{1794}{3} = 598$$

Thus 598 three digit numbers starting from 100 will taken into account. i.e., up the number 697. The 1983^{rd} digit is therefore 7. i.e., $q = 7$ which is required.

1.07 If α is a common root, then,

$$1998\alpha^2 + a\alpha + 8991 = 0$$

$$8991\alpha^2 + a\alpha + 1998 = 0\,.$$

Subtracting the second equation from the first.

$6993\alpha^2 - 6993 = 0\,. \quad \therefore \quad \alpha^2 = 1$ or $\alpha = \pm 1\,.$

If $\alpha = 1$, then $a = -8991 + 1998 = -10989\,.$

If $\alpha = -1$, then $a = 1998 + 8991 = 10989\,.$

Thus $a = \pm 10989\,.$ Therefore, for the value $a = \pm 10989$, the equations have a common root.

1.08 Since x, y go with an even power, it suffices to consider non-negative values for x, y. Since 3 does not divide $1987, x$ cannot be 0. y cannot be obviously zero. If x is a natural number, x^4 has for its last digits the numbers $1, 5$ or 6. So $3y^4$ has for its last digits one of $3, 8, 5$ only (RHS). Now $2x^4$ has for its last digits one of $2, 0$.

$\therefore \quad 2x^4 + 1987$ has for its last digit $2 + 7 = 9$ or $0 + 7 = 7$ (LHS). But LHS and RHS do not agree in the last digit. Thus the equation cannot be satisfied for any natural number values of x, y and so for any integer values.

$z > y > x \geq 1$ so that $y > 1\,.$

1.09 The equation is equivalent to $19xyz = 97(yz - z - 1)$ or $19xyz + 97 = 97z \times (y-1)$. (A)

19 and 97 are primes and so 97 should divide one of x, y, z. Also z should divide 97 which means that $z = 97$ is prime. (B)

Then $97(19xy+1) = 97^2(y-1)$ or $19xy+1 = 97(y-1)$. i.e., $19xy = 97y - 98$.

From this it follows that y divides $98 = 2 \times 49 = 2 \times 7^2$ y being less than $z = 97$ The possibilities are $y = 2$, $y = 7$, $y = 14$, $y = 49$. (C)

If $y = 2, 19 \times 2 \times x = 2 \times 97, 98 = 96$ so that x is not an integer.
If $y = 7$, $19x = 83$ so that x is not an integer.
If $y = 14$, $19x = 90$ so that x is not an integer.
If $y = 49, 19 \times 49x = 49 \times 97 - 98$.
i.e., $19x = 57 - 2 = 55$ so that $x = 5$. (D)
Thus $x = 5, y = 49, z = 97$.

1.10 Adding the three equations ,we get
$2(x+y+z)^2 = 242$ or $(x+y+z)^2 = 121$

$$\therefore \quad x+y+z = \pm 11 \tag{A}$$

Using this with the given equations, $x + y = \pm 6$; $y + z = \pm 9$; $z + x = \pm 7$. Thus solving $x = \pm 2$, $y = \pm 4$, $z = \pm 5$

$\therefore (x, y, z) = (2, 4, 5)$ and $(-2, -4, -5)$. (+ yielding one set of solutions and $-$ the other set). Thus there are 2 sets of values for x, y, z.

1.11 $N_K = 13\ 13\ 13\ 13\ldots 131$ be the $(2k+1)$ the digit number in base 10, formed by K pieces of 13 and appended by 1 at the end.

Now $N_k = 31+(31\times 10^2)+(31\times 10^4)+\ldots+(31\times 10^{2k-1})+10^{2k}$ so that 31 divides N_K if and only if it divides 10^{2k}. 31 is a prime different from 2 and 5 which are prime factors of 10^{2k}. Hence 10^{2k} is not divisible by 31 and so N_k also.

1.12 Now $13 = 3+3+3+4 = 5+5+3 = 4+4+5$ these being the only ways of partioning 13 using only $3, 4, 5$. Let x rods be cut in the first way, y in the second way and z in the third way. Under the given condition there should be 13 pieces each of length $3, 4, 5$ units. So, $3x + y = 13$, $x + 2z = 13$, $2y + z = 13$.

We have to solve for x, y, z from these equations. Eliminating x from the first equations. $6z - y = 26$ which gives along with the third equations $13z = 65$ or $z = 5$ so that $y = 4$.

From the second equations, $x = 3$. Thus 3 rods are to be cut to pieces of $3, 3, 3, 4$ units, 4 rods into pieces of $5, 5, 3$ units and 5 rods in to pieces of $4, 4, 5$ units.

1.13 If both a and b are even, then, from the first equation, the product ab ends in 1 or 6.

However, in this case, a, b cannot have 1 as unit digit. For a, b to end in 6, each of a, b should be at least 6 but then $5a + 5b + 2ab = 30 + 30 + 72 > 92$. Thus both a, b cannot be even. If one of them is even and the other odd, $5a + 5b + 2ab$ will be odd and cannot be equal to 92.

Thus, both a and b are to be odd, if at all. In this case also a, b has to have 1 or 6 as last digit. But a, b being odd, it can only be 1.

$a = b = 1$ is not possible and $(1, 11)$ also not possible, then next choice is $a = 3, b = 7$ which satisfies the first equation.
Substituting in second and third equations.

$$35 + 5c + 14c = 136$$

$$19c = 101$$

which is not possible since 'c' is a positive integer. Also $5c + 15 + 6c = 11c + 15 = 244$.
So that $11c = 229$ which is not possible.

On the other hand, if $a = 7$ and $b = 3$,

then $15 + 5c + 6c = 11c + 15 = 136$

i.e, $11c = 121 \to c = 11$ and $5c+35+14c = 19c+35 = 244$ i.e., $19c = 209$. Giving $c = 11$, thus $a = 7$, $b = 3$, $c = 11$ is a solution.

For the first equation, possible pairs like $(9, 9)$ cannot be solutions since they give a value larger than 92 for the LHS. Thus the solution arrived at for a, b, c is the unique solution. With this solution, $7a + 8b + 9c = 172$.
(Adapted from solution by sriharsha sista)

1.14 If c^3 is a square, then $c = c_1^2$ and if a^5 is a 4^{th} power, then, $a = a_1^4$, then $c_1^2 - a_1^4 = 19$.

i.e., $(c_1 - a_1^2)(c_1 + a_1^2) = 19$, both the factors being positive integers. 19 being a prime, $c_1 - a_1^2 = 1$ and $c_1 + a_1^2 = 19$

so that $c_1 = 10$ and $2a_1^2 = 18$ or $a_1 = 3$.

$d^2 = c^3 = (c_1)^2 = c_1^6 \therefore d^2 = 10^6$.

Hence $d = 10^3; b = 3^5; d - b = 10^3 - 3^5$.

1.15 The exact sum is $\dfrac{x(x+1)}{2}$ i.e., $\dfrac{x^2+x}{2}$.

Now $\dfrac{x(x+1)}{2} < 1998$ i.e., $x^2 + x < 3996$.

Also $x^2 < x(x+1) < (x+1)^2 \therefore 63^2 < 3996 < 64^2$

Also $63 \times 64 > 3996$;

$63 \times 63 < 3996 < 63 \times 64$.

Thus $x = 62$. $\therefore$ $\dfrac{x(x+1)}{2} = 1953$ with $x = 62$.

Hence the page added twice is $1998 - 1953 = 45$.

1.16 The *l.c.m* is least when all factors are equal so that this number is the *l.c.m*. Since the sum is 801, all the 20 numbers cannot be equal. So 19 numbers are equal to 40 and the 20th number is 41. yielding the sum $19 \times 40 + 41 = 801$.

Hence the least l.c.m is $40 \times 41 = 1640$.

1.17 If the 3-element subsets in the class are pairwise disjoint then one of the subsets has less number of elements and so at least one pair has one element in common but not two or more. If just one pair of subsets has common elements,they have to have two elements in common. so, this is also not possible. Thus the problem reduces to finding a class of all 3-element subsets with precisely one element common between any two of the subsets.

Clearly $A = (a,b,d)$, (b,c,e), (c,d,f), (d,e,g), (e,f,a), (a,c,g), (b,f,g) is one such class .Another possible collection is

$B = \{a,b,c\}, \{a,d,e\}, \{a,f,g\}, \{b,d,f\}, \{b,e,g\}, \{c,e,f\}$ and $\{c,d,g\}$. Any permutation of A will give another class.

1.18: Let x be the length and y the breadth of the field.

$$\therefore \quad 2x + 2y = 3996 \text{ i.e., } x + y = 1998. \tag{1}$$

Also area xy is exactly divisible by 1998 (2)

i.e., $x(1998 - x)$ is exactly divisible by 1998. (3)

Now $1998/x^2$ and $x \geq \dfrac{1998}{2} = 999.$ (4)

Since $1998 = 2 \times 9 \times 3 \times 37$ divides x^2 the least value for x is $2 \times 9 \times 37 = 666$ metres. But since x is the length of the field $x > 999$ and hence $x = 666 \times 2 = 1332$ and $b = 666$ metres.

$$\therefore \quad \text{length } = 1332 \text{ feet, breadth } = 666 \text{ feet.}$$

1.19: Since the only possible integer root of $x^5+2x+1=0$ are ± 1 and these are not the zeroes of the polynomial, it has no linear factors. So we can express,
$x^5 + 2x + 1 = p(x)q(x)$ where p is of degree 2 and q of degree 3. Since coefficient of x^5 is 1, we can assume without loss of generating $p(x) = x^2 + ax + b$ and $q(x) = x^3 + cx^2 + dx + e$. Since $be = 1$, either $b = e = 1$ or $b = e = -1$.

Case1: Let $b = e = 1$. Comparing $x-$terms on both sides, we get $ae + bd = 2$.

i.e., $a + d = 2$ (i)

and comparing the coefficients of x^4 and x^3,

we get $a + c = 0 \quad \therefore \quad c = -a$

$d + ac + b = 0 \quad \therefore \quad b + d = a^2 \quad \therefore \quad a^2 = d + 1$. (ii)

We get from (i) and (ii), $a^2 - 1 + a = 2 . a^2 + a - 3 = 0$.

This clearly has no integer roots. (*No solution*).

Case2: Let $b = e = -1$.

$\therefore \quad a + d - 2$ and $b + d = a^2$ and $a^2 = d - 1$

$a + a^2 + 1 = 2$ i.e., $a^2 + a + 3 = 0$ It has also no integer roots.

Thus there is no solution meaning there do not exist polynomials $p(x)$ and $q(x)$, each having integer coefficients and of degree greater than or equal to 1 such that

$$p(x)q(x) = x^5 + 2x + 1.$$

1.20: Let $0 < \alpha < \beta < \gamma < \delta$ be the four real positive (distinct) numbers.

Now, $\alpha\beta < \alpha\gamma < \beta\gamma < \beta\delta < \gamma\delta.$ (1)

Now, for an equation $px^2 + x + q = 0$. We have real roots we must have $1 - 4pq \geq 0$ i.e., $pq \leq 1/4$.

So,the equation has imaginary roots or real roots depending on $pq > 1/4$ or $pq \leq 1/4$.

Let $\beta\gamma > 1/4$. Even $\beta\delta, r\delta$ are also $> 1/4$.

Choose $\beta = A, \gamma = B$ and $\delta = C$. We get the required

three equations whose roots are complex conjugates. If, on the other hand, $\beta\gamma \leq 1/4$, then $\alpha\beta$, $\alpha\gamma$ are both $< 1/4$.

Choose $A = \alpha$, $B = \beta$ and $C = \gamma$. Then all the three equations have real roots. We can choose 3 numbers out of the given 4 numbers satisfying the given conditions.

Note: If the positive condition is removed, we can give a counter example for the given statement of the problem.

$\alpha = -2$, $\beta = -1/4$, $r = 1, \delta = 2$

$1 - 4\gamma\delta = 1 - 8 = -7 < 0$, $1 - 4\beta\delta = 3 > 0$

$1 - 4\beta\gamma = 2 > 0, 1 - 4\alpha\gamma = 9 > 0, 1 - 4\alpha\beta = -1 < 0.$

Note here more of the combinations (α, β, γ), (α, γ, δ), (α, β, δ), (β, γ, δ) satisfy the condition of the problem.

1.21: The sum of the elements of each subset if divisible by 4.So the sum of all 4 elements = sum of the n sum of the 4 element subsets is also divisible by 4.

Sum of the 4 elements $= \dfrac{4n(4n+1)}{2} = 2n(4n+1)$ is divisible by 4. Thus implies that n is even i.e., $n = 2k$. Hence the number of elements for such a partition to be possible is $8k$.

To Prove the Existence of such a partition:

Consider $K = 1$, we can partition $\{1, 2, 3, \ldots 8\}$ into sets $\{4, 3, 1, 8\}$ and $\{5, 2, 7, 6\}$.

Given any $\{8n + 1, 8n + 2, \ldots 8n + 8\}$, we can partition it as $\{8n+4, 8n+3, 8n+1, 8n+8\}$ and $\{8n+5, 8n+2, 8n+7, 8n+6\}$.

Hence, when the number of elements is $8k$, such a partitioning is possible.

1.22: Let the letters be numbered $1, 2, 3, 4$ and their correct envelopes be A, B, C, D. Then the number of ways of not putting 1 in A. (by putting 1 in B) is shown below.

A	B	C	D	
4	1	2	3	
4	1	3	2	✓
3	1	2	4	✓
3	1	4	2	
2	1	4	3	
2	1	3	4	✓

The second row 4132 does not satisfy the conditions as 3 is put in the correct envelope. Similarly the 3rd and 6th rows do not satisfy the conditions. Thus in total the number of ways of not putting any of the letters in their correct envelopes by putting 1 in B is 3.

Similarly when C has 1, there are 3ways for not putting any of the letters in the correct envelope and similarly for D.

$\therefore$ The total number of ways by which no one gets right envelopes is $3 + 3 + 3 = 9$.

1.23: Let the number of elements in C be x. Then the number of elements in B is $(x + y)$ for some y and the number of elements in A is $2(x + y)$.

Then the number of subsets in C is 2^x and of B is 2^{x+y}.

$$\therefore \quad 2^{x+y} - 2^x = 15 \Rightarrow 2^x(2^y - 1) = 15.$$

Here 2^x is odd $x = 0$ and $2^y - 1 = 15$ gives $y = 4$. Hence the number of subsets of A is 2^8.

Hence number of subsets of A exceeding number of subsets of B is $2^8 - 2^4 = 240$.

1.24: $x^2 + x + 1 = 0$.

$$\therefore \quad x^2 = -(x+1) \tag{1}$$

$$x^{1999} + x^{2000} = x^{1999}(x+1) \tag{2}$$

$$x(x+1) = -1 \tag{3}$$

Now, $x^{1999} = -x^{1997}(x+1)$ (from (1))

$x^{1999} + x^{2000} = -x^{1997}(x+1)^2$ (from (2))

$x^{1999} + x^{2000} = x^{1995}(x+1)^3$

So, $x^{1999} + x^{2000} = -x^{1997}(x+1)^2 = x^{1995}(x+1)^3 =$

$-x^{1993}(x+1)^4 = x^{1996}(x+1)^5$ and so on till we get,

$x^{1999} + x^{2000} = x^{999}(x+1)^{999} = [x(x+1)]^{999} = -1.$

1.25: The common points of the graphs are given by the solution of the equations, $y = x^2 - 6x + 2, x = 4 - 2y$.

Thus $y = (4 - 2y)^2 - 6(4 - 2y) + 2$.

i.e., $(4y + 3)(y - 2) = 0$.

On simplification $\therefore y = -3/4$ or 2. $\therefore x = 11/2$ or 0. $\therefore A$ is $(11/2, -3/4)$ and B is (0,2). Mid point of AB is $(11/4, 5/8)$.

1.26:

$$\left[\frac{1}{2}\right] < 1; \left[\frac{1}{2} + \frac{1}{100}\right] < 1; \left[\frac{1}{2} + \frac{2}{100}\right] < 1 \ldots \left[\frac{1}{2} + \frac{49}{100}\right] < 1.$$

But after this,

$$\left[\frac{1}{2} + \frac{50}{100}\right] = 1; \left[\frac{1}{2} + \frac{51}{100}\right] > 1 \left[\frac{1}{2} + \frac{52}{100}\right] > 1 .$$

Since $\frac{1}{2} + \frac{51}{100} < 2$ and so on till $\left[\frac{1}{2} + \frac{61}{100}\right] = 1$.

i.e., 1 occurs 12 times in the given sum. Hence

$$\left[\frac{1}{2}\right] + \left[\frac{1}{2} + \frac{1}{100}\right] + \left[\frac{1}{2} + \frac{2}{100}\right] + \ldots + \left[\frac{1}{2} + \frac{61}{100}\right]$$
$$= 0 + 0 + 0 \ldots + 0 + 1 + 1 + \ldots + 1.$$

$\therefore$ Given sum $= 12 \times 1 = 12$.

1.27: Let a boy take b days to do the work and a man take m days. Then 1 boy does $1/b$ of the work and a man does $\frac{1}{m}$ of the work in a day. then 3 men an 5 boys in 3 days do

$$3\left(\frac{3}{x} + \frac{5}{b}\right) \text{ of the work } = \frac{19}{20}. \tag{1}$$

Similarly 4 men and 8 boys do in 2 days

$$2\left(\frac{4}{m}, + \frac{8}{b}\right) \text{ of the work } = \frac{14}{15} \tag{2}$$

$$\text{i.e., } \frac{72}{m} + \frac{120}{b} = \frac{152}{20} \tag{3}$$

$$\frac{72}{m} + \frac{144}{b} = \frac{126}{15} \tag{4}$$

Solving (3)and (4).We get $b = 30$.

$\therefore$ *A* boy takes 30 days to finish the whole work.

1.28: If $x^2 - y^2 = 31$, then $(x+y)(x-y) = 1 \times 31$.

Since x, $y \in N, 31$ being prime, $x + y = 31$, $x - y = 1$ yielding $x = 16$; $y = 15$.

1.29: The number of words (different) that can be formed with 4 different letters of the alphabet where any letter may repeat itself is $4^k = 256$.

Similarly, words with 3 letters, 2 letters and one letter are respectively. 64, 16 and 4.

So, the number of words cannot exceed $256+64+16+4 = 340$.

1.30: Let the number of successful and the unsuccessful competitors be x and y respectively. Then average jump length

$$= \frac{\text{total length jumped by all the participants}}{\text{number of competitions}}$$

$$= \frac{(6.5x + 4.5y)}{x+y} 4.9.$$

Hence $\dfrac{x}{y} = \dfrac{1}{4}$. $\therefore \dfrac{x}{x+y} = \dfrac{1}{5}$ and $\dfrac{1}{5}$ is 20%.

$\therefore$ Percentage of successful competitions $= 20\%$.

1.31: The equations can be written as

$\dfrac{1}{x} + \dfrac{1}{y} = 5, \dfrac{1}{y} + \dfrac{1}{z} = 6, \dfrac{1}{z} + \dfrac{1}{x} = 7$, if x, y, z are non-zero.

$\therefore \quad \dfrac{1}{z} - \dfrac{1}{x} = 1$ and $z = \dfrac{1}{4}$

$\therefore \quad x = \frac{1}{3}$ (Subtracting 1st equation from the second equation) $\therefore \quad y = \frac{1}{2}, \ x + y + z = \frac{13}{12}$.

1:32: $2x + 5y = 100$.

$$\therefore \quad x = \frac{100 - 5y}{2}. \qquad \text{(A)}$$

Since $x \in N, y$ must be even. (B)

Since $x \in N, y < 20$. (C)

(Otherwise $(100 - 5y)$ will be negative.)

There are 9 natural numbers for y (even natural numbers between 1 and 20.)There are 9 values for x correspondingly. The total number of (x, y) pairs is therefore 9.

1.33: $2^3 < 3^2 < 10$

$\therefore \quad 2^9 < 3^6 < 1000$. Compare $(1.0.1)^{1000}$ and 1000.

If x is any integer > 1 and $h > 0$, $(1 + h)^x > 1 + xh$.

$(1 + 0.01)^{100} > 1 + 100(0.01) = 2$

Also $(1.001)^{1000} > 1 + 1000 \times .001 = 1 + 1 = 2$.

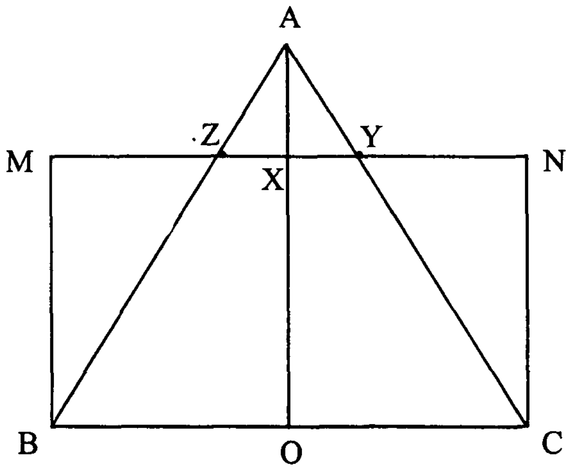
Then $(1.01)^{1000} > 2^{10} > 10.00$.

$\therefore \quad$ The greatest number is $(1.01)^{1000}$.

According order is

$$(1.001)^{1000} < 2^9 < 3^6 < 1000 < (1.01)^{1000}.$$

1.34:



If all the three vertices lie in two unit squares, then at least two of the vertices lie in one unit square. But these vertices are 2 units apart while the maximum distance between two points in the unit square is $\sqrt{2} < 2$ .There is a contradiction. So two squares are not enough. From the figure it is clear that 3 squares are enough.

1.35: $2^{15} + 2^{15} = 2.2^{15} = 2^{16}$.

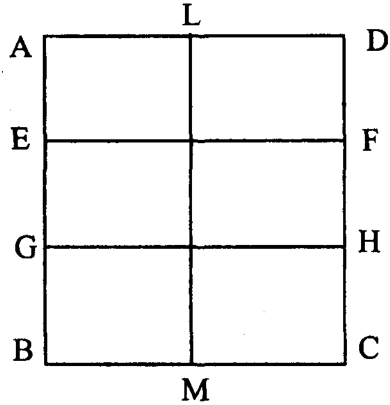
$2^{15a} + 2^{15a} = 2 \times 2^{15a} = 2^{15a+1} = 2^{14a+a+1}$.

Thus, if we choose a such that $7\backslash(a+1)$, i.e., if we choose $a = 7k - 1, k = 1, 2, 3\ldots$ we have,

$\left(2^{5(7k-1)}\right)^3 + \left(2^{3(7k-1)}\right) = \left(2^{(15k-2)}\right)^7$

i.e., $x = 2^{5(x-1)}, y = 2^{3(7k-1)}, z = 2^{15k-2}, k = 1, 2, \ldots$ are the infinitely many triples giving the solutions.

1.36:



$ABCD$ is a square of side 1. Draw EF and GH parallel to AD and draw LM parallel to AB so that $AE = EG = GB = \frac{1}{3}$ and $AL = LD = \frac{1}{2}$. Now the square has been divided into six congruent rectangles each having a diagonal of length $\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{9}} = \frac{\sqrt{13}}{6}$.

By the pigeon-hole principle, at least two of the seven points will be inside on the bonder of one of the six rectangles i.e., at least two of the 7 points are at a distance not greater than $\frac{\sqrt{13}}{6}$ which is the length of a diagonal of a rectangle.

1.37: Now $(19P + 99Q) + (19Q + 99P)$

$$\begin{aligned}
&= (19 + 99)P + (99 + 19)Q = 118P + 118Q \\
&= 118(P + Q) = 118(ab + bc + cd + ac + ad + bd) \\
&= 118\left\{\frac{(a + b + c + d)^2 - (a^2 + b^2 + c^2 + d^2)}{2}\right\} \\
&= 59(0 - (a^2 + b^2 + c^2 + d^2)) \text{ as } a + b + c + d = 0 \text{ (given)} \\
&= -59(a^2 + b^2 + c^2 + d^2)
\end{aligned}$$

which is a negative quantity. $\therefore$ At least one of $19P+99Q$ and $19Q+99P$ must be negative.

1.38: Any multiple of 15 has to end with 0 or 5. But 5 is excluded by hypothesis of the problem. So the last digit of the multiple has to be 0.

Also the multiple has to be divisible by 3 and so the sum of the digits of the multiple has to be divisible by 3. As the sum of the digits of the multiples entirely depends on 8. The contribution of 0 being nil, the minimum number of digits 8 in the number will be 3 so as to make it divisible by 3. Now 8880 is divisible by 3 as well as 5 i.e., divisible by 15. Hence 8880 is the required smallest number.

Note: Insertion of zeroes in between the digits which are 8 makes the number larger.

1.39: As p, q, r are the roots of the equations

$x^3 - 3px^2 + 3q^2x - r^3 = 0$, we have the identity

$$x^3 - 3px^2 + 3q^2x - r^3 \equiv (x-p)(x-q)(x-r)\,. \tag{1}$$

Equating the coefficients of identical pairs of x,

$$p+q+r = 3p \tag{2}$$

$$pq+qr+rp = 3q^2 \tag{3}$$

$$pqr = r^3\,. \tag{4}$$

Now, if $p = 0$, then $r = 0$ by (4) $\rightarrow q = 0$ by (1) i.e., if $p = 0$, then $p = q = r = 0$. Also if $q = 0$, then also $r = 0$ by (4) and $p = 0$ by (2). i.e., if $q = 0$, then $p = q = r = 0$.

Finally if $r = 0$, then (3) implies $pq = 3q^2 \rightarrow q(p-3q) = 0$ $\Rightarrow q = 0$ or $p = 3q$.

In both the cases, by (2), we have $p = q = r = 0$. Thus we see that, if any one of p, q, r is 0, we have $p = q = r = 0$. So we can suppose that none of p, q, r is 0.

Then (4) $\rightarrow pq = r^2$ (5)

So (3) implies that $r^2 + r(p+q) = 3q^2$

$$\Rightarrow r(r+p+q) = 3q^2$$

$$\Rightarrow 3pq = 3q^2$$

$$\Rightarrow pr = q^2 \qquad (6)$$

Now (5) and (6)

$$\Rightarrow \frac{pq}{p^2} = \frac{r^2}{q^2} \Rightarrow \frac{q}{r} = \frac{r^2}{q^2} (\text{as } p \neq 0)$$

$$\Rightarrow q^3 = r^3 \Rightarrow q^3 - r^3 = 0$$

$$\Rightarrow (q-r)(q^2 + r^2 + qr) = 0$$

$$\Rightarrow q - r = 0 \text{ (as } q^2 + r^2 + qr > 0, \Rightarrow \text{ for } p, q, r > 0)$$

$$\Rightarrow q = r \qquad (7)$$

Now (5) and (7) will give $pr = r^2 \Rightarrow p = r$ as $r \neq 0$. (8)
Hence (7)and (8) will give $p = q = r$.

1.40: Let $u = m + n; v = m - n$. Now,

$$v^2 = \frac{u^2 - v^2}{u - 1}$$

$$\therefore v^2 u - v^2 = u^2 - v^2$$

$$u(v^2 - u) = 0$$

So either $u = 0$ or $u = v^2$. Since $u \neq 1$ and $m = \frac{u+v}{2}$, $n = \frac{u-v}{2}$. The Solutions are given by $\{(m, -m) : m \in Z\}$;

$$\left\{\left(\frac{v(v+1)}{2}\right), \frac{v(v-1)}{2} : v \in Z, |v| \geq 2\right\}$$

1.41: Let each of the given ratios be equal to k.

Then

$$bx + c - cx = ka$$

$$cx + a - ax = kb$$

$$ax + b - bx = kc$$

Adding we get $a + b + c = k(a + b + c)$

Hence $a + b + c = 0$ or $k = 1$ (A)

If $K = 1$, then multiplying the above equations by a, b, c respectively.

$$abx + ca - cax = a^2$$

$$bcx + ab - abx = b^2$$

$$acx + bc - bcx = c^2$$

Adding $ab + bc + ca = a^2 + b^2 + c^2$

$\therefore \quad (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$.

i.e., $a = b = c$.

1.42: $a^n + a^n + \ldots + a^n = a^{n+1}$ given $\Rightarrow m.a^n = a^{n+1}$ $\Rightarrow m = a$ Similarly $n = b$.
Therefore $mn - (ab - 1) = mn - ab + 1 = 1$.

1.43: The only values satisfying $a^2 + b^2 = 25$ are $a = 3, b = 4$ or $a = 4, b = 3$ with $a + b = 7$ in both cases. Similarly $x^2 + y^2 = 13$ is satisfied by $x = 2, y = 3$ or $x = 3, y = 2$ with $x + y = 5$ in both cases. Thus

$$\begin{aligned} ax + by + ay + bx &= (a+b)(x+y) \\ &= 7 \times 5 = 35. \end{aligned}$$

1.44: In the sequence of natural numbers.

$1, 1, 2, 2, 3, 3, \ldots r, r, (r+1), (r+1)$

$t_{2n-1} = t_{2n} = n$ where t_n is the n^{th} term of the sequence.

$$\begin{aligned} \therefore \quad f(2n) &= t_1 + t_2 + \ldots + t_{2n} \\ &= 1 + 1 + 2 + 2 + \ldots + n + n \\ &= 2\frac{n(n+1)}{2} = n(n+1) \\ \text{and } f(2n-1) &= f(2n-2) + t_{2n-1} \\ &= f(2(n-1)) + t_{2n-1} \\ &= (n-1)n + n = n^2 \end{aligned}$$

Now k is an odd number say $k = 2n - 1$. Then $k^2 = 4n^2 - 4n + 1 = 2(2n^2 - 2n + 1) - 1$ and $n = \dfrac{k+1}{2}$.

$\therefore \quad f(k^2) = (2n^2 - 2n + 1)^2$.

But $n = \dfrac{k+1}{2}$.

$$\begin{aligned} \therefore \quad f(k^2) &= \left[2(\frac{k+1}{2})^2 - \frac{2(k+1)}{2} + 1\right]^2 \\ &= \left[\frac{(k+1)^2}{2} - (k+1) + 1\right]^2 \end{aligned}$$

$$= \frac{1}{4}\left[(k+1)^2 - 2(k+1) + 1 + 1\right]$$

$$= \frac{1}{4}[k^2+1]^2 \text{ or } \left(\left(\frac{k^2+1}{2}\right)\right)^2$$

1.45: As n has four zeroes at the end, it should have 5^4 as a factor. Hence n can be $20, 21, 22, 23$, or 24. Further $(n+1)!$ must have six squares at the end. This is possible if $n+1 = 25$ or $n = 24$.

1.46: $2x^2 + 3y^2 = 35$, $x, y \in z$. Clearly y cannot be an even integer and $|y| \leq 3$. (A)

Thus y can take values $\pm 1, \pm 3$.

The corresponding x values are $\pm 4, \pm 2$. Thus the only ordered pairs (x, y) satisfying the equation are (4,1), $(-4,1)$, $(4,-1)$, $(-4,-1)$, (2,3), $(2,-3)$, $(-2,3)$, $(-2,-3)$.

1.47: Let the four numbers be a, b, c and d. Thus

$$a + b + c + 3d = 51 \tag{1}$$

$$a + b + 3c + d = 63 \tag{2}$$

$$a + 3b + c + d = 69 \tag{3}$$

$$3a + b + c + d = 87 \tag{4}$$

$$\therefore \quad a + b + c + d = \frac{270}{6} = 45 \tag{5}$$

$\therefore$ $2d = 6$ or $d = 3, 2c = 18$ or $c = 9, 2b = 24$ or $b = 12$, $2a = 42$ or $a = 21$.

1.48: Since there are 7 different colours and we need two balls of the same colour, by pigeon-hole principle, we should collect at least $(7+1) = 8$ balls.

Note: In general, if there are balls of k different colours and we want n balls of the same colour chosen, then, we should collect. $\{(n-1)K+1\}$ balls.

1.49: Sum of the squares of the roots,

$$= (a-2)^2 - 2(1-a) = (a-1)^2 + 1\,.$$

The minimum of the expression $(a-1)^2+1$ is 1 when $a-1=0$. Thus $a=1$ for the least possible value of the sum of the squares of the roots.

1.50: As $\alpha - \beta = 8$ or $\beta - \alpha = 8$, we have either $\alpha = 7, \beta = -1$ or $\alpha = -1, \beta = 7$. In both cases, $\alpha\beta = -7$. The equation whose roots are α and β is $x^2 - (\alpha+\beta)x + \alpha\beta = 0$ or $x^2 - 6x - 7 = 0$.

1.51: $F(3, f(4)) = F(3,2) = 2^2 + 3 = 7$. $(\because\ f(4) = 2)$.

1.52:

$$(10a+b)(10a+c) = 100a(a+1) + bc$$

$$\begin{aligned} \Rightarrow 100a^2 + 10a(b+c) + bc &= 100a^2 + 100a + bc \\ &= 100a^2 + (10a \times 10) + bc \\ \Rightarrow b + c &= 10. \end{aligned}$$

1.53: Let $x = K_1 y^3$ and $y = K_2 z^{\frac{1}{5}}$.

$$\begin{aligned} \text{Then } x &= K_1(K_2 z^{\frac{1}{5}})^3 \\ &= (K_1K_2) z^{\frac{3}{5}} \\ &= K z^{\frac{3}{5}} \\ \therefore\quad x &= \frac{3}{5} \end{aligned}$$

1.54: As $2001 = 3 \times 23 \times 29$ and the numbers used are even (i.e.,) $2 \cdot 4 \cdot 6 \cdot 8 \ldots$, the product must include $6, 46, 58$. The largest even integer used must be 58.

1.55: Divisibility by $72 \Rightarrow$ divisibility by 8 and 9.Divisibility by $8 \Rightarrow$ last 3 digits must be divisible by 8. $\Rightarrow b = 2$ in $a679b$. Divisibility by $9 \Rightarrow a+6+7+9+2$ is divisible by 9. $\Rightarrow a = 3$. Number is 36792.

1.56: My maximum age is $1 + 9 + 9 + 9 = 28$. Let the year of birth be $19xy$.

$$\therefore\ 2000 - 19xy = 10 + x + y$$
$$\therefore\ 100 - xy = 100 - 10x - y = 10 + x + y$$
$$\therefore\ 90 = 11x + 2y (1 \leq x \leq 9, 1 \leq y \leq 9)$$
$$\therefore\ 11x = 90 - 2y \geq 90 - 18 = 72$$

$\therefore \quad x > \dfrac{72}{11}$ i.e., $x \geq 7$ (x is an integer) and x is even as $(11x + 2y)$ is even

$\therefore \quad x = 8, 2y = 90 - 88 = 2 \quad \therefore \quad y = 2$

$\therefore$ Year of birth is 1981.

1.57: $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b)$ a, b are two-digit prime $\Rightarrow$ they are odd.

Both a and b are of the $4k \pm 1$.

Case(i): Let $a = 4k + 1, b = 4l + 1$,

$$\begin{aligned} \therefore\quad a - b &= 4(k - l) \Rightarrow (a - b) \text{ is divisible by } 4 \\ a + b &= 4(k + l) + 2 \Rightarrow (a + b) \text{ is divisible by } 2 \end{aligned}$$

$$\begin{aligned} a^2+b^2 &= (4k+1)^2+(4l+1)^2 \\ &= 16(k^2+l^2)+8(k+l)+2 \\ &\Rightarrow (a^2+b^2) \text{ is divisible by 2.} \end{aligned}$$

Thus (a^4-b^4) is divisible by 16. (A)

Case(ii): Let $a=4k+3, b=4l+3$.

a^4-b^4 is divisible by 16 similarly. (B)

Case (iii): Let $a=4k+1, b=4l+3$.

$\therefore \quad a+b=4(k+l+1) \Rightarrow (a+b)$ is divisible by 2.

$a^2+b^2=$ even $\Rightarrow a^2+b^2$ is divisible by 2.

Thus a^4-b^4 is divisible by 16. (C)

Case (iv): The case $a=4k+3, b=4l+1$ is similar. (D)

Also $\left.\begin{aligned} a^2 &\equiv 1(\text{mod}3) \\ b^2 &\equiv 1(\text{mod}3) \end{aligned}\right\} \Rightarrow \left.\begin{aligned} a^4 &\equiv 1(\text{mod}3) \\ b^4 &\equiv 1(\text{mod}3) \end{aligned}\right\}$

$\Rightarrow a^4-b^4 \equiv 0(\text{mod}3)$. (E)

$\therefore \quad a^4-b^4$ is divisible by 3. (F)

Also $\left.\begin{aligned} a &\equiv 1(\text{mod}5) \\ a &\equiv 2(\text{mod}5) \\ a &\equiv 3(\text{mod}5) \\ a &\equiv 4(\text{mod}5) \end{aligned}\right\} \Rightarrow a^4 \equiv 1(\text{mod}5)$ (G)

Similarly $b^4 \equiv 1(\text{mod}5)$ (H)

$\therefore \quad a^4-b^4 \equiv 0(\text{mod}5)$ (I)

$\therefore \quad a^4-b^4$ is divisible by 5 (J)

Thus (a^4-b^4) is divisible by 16,3,5 i.e., 240

i.e., $240/(a^4 - b^4)$ (K)

To prove the second part, consider $a = 13$, $b = 11$.

Then $a^4 - b^4 = (168 + 121)(24)(2) = 2^4 \times 3 \times 5 \times \times 58$ (L)

For $a = 17, b = 13, a^4 - b^4 = (289 + 165)(30)(4)$

$$= 2^4 \times 3 \times 5 \times 229 \quad \text{(M)}$$

229 is co-prime with 58and 229 is itself prime

$\therefore$ g.c.d of $a^4 - b^4$,

$\therefore$ $a = 13, b = 11$ or $a = 17, b = 13$ is 240.

Hence 240 is the g.c.d of all such numbers $a^4 - b^4$ where a, b are two-digit primes.

1.58: $f(x) = \dfrac{16^x}{16^x + 4} = \dfrac{(16^x + 4) - 4}{16^x + 4} = 1 - \dfrac{4}{16^x + 4}$ (A)

$$f\left(\frac{K}{2000}\right) + 6\left(\frac{2000 - k}{2000}\right)$$

$$= \left(1 - \frac{4}{(16^{\frac{k}{2000}} + 4)}\right) + \left(1 - \frac{4}{(16^{\frac{2000-k}{2000}} + 4)}\right)$$

$$= 2 - 4\left[\frac{8 + 16^{\frac{k}{2000}} + 16^{\frac{2000-k}{2000}}}{16 + 4(16^{\frac{k}{2000}} + 16^{\frac{2000-k}{2000}}) + 16}\right]$$

$$= 2 - 4\left[\frac{8 + 16^{\frac{k}{2000}} + 16^{\frac{2000-k}{2000}}}{4(8 + 16^{\frac{k}{2000}} + 16^{\frac{2000-k}{2000}})}\right] = 1$$

$$\therefore \sum_{K=1}^{1999} f\left(\frac{k}{2000}\right) = \sum_{k=1}^{999}\left[f\left(\frac{k}{2000}\right) + f\left(\frac{2000 - k}{2000}\right)\right] + f\left(\frac{1}{2}\right)$$

$$= \sum_{k=1}^{999} 1 + \frac{\sqrt{16}}{\sqrt{16} + 4} = 999 + \frac{4}{8} = 999\frac{1}{2}.$$

1.59: The following is one arrangement.

91,	92,	93,	94,	95,	96,	97,	98,	99,	100,
81,	82,	83,	84,	85,	86,	87,	88,	89,	90,
71,	72,	73,	74,	75,	76,	77,	78,	79,	80,
...				...				...	
11,	12,	13,	14,	15,	16,	17,	18,	19,	20,
1,	2,	3,	4,	5,	6,	7,	8,	9,	10

Another arrangement:

10,	9,	8,	7,	6,	5,	4,	3,	2,	1,
20,	19,	18,	17,	16,	15,	14,	13,	12,	11,
30,	29,	28,	27,	26,	25,	24,	23,	22,	21,
...				...				...	
100,	99,	98,	97,	96,	95,	94,	93,	92,	91

1.60: One solution is $x = y = z = 0$. To find other solutions, if any, let $x^2 + y^2 + z^2 = 2xyz$

If none of x, y, z is even, then

$$x^2 + y^2 + z^2 \equiv (1 + 1 + 1)(\bmod 4), 2xyz = 2(\bmod 4)$$

If exactly one of them is even, then

$$x^2 + y^2 + z^2 \equiv (0 + 1 + 1)(\bmod 4); 2xyz = 0(\bmod 4)$$

If two of x, y, z are even and one is odd,

$$x^2 + y^2 + z^2 \equiv (0 + 0 + 1)(\bmod 4); 2xyz = 0(\bmod 4)$$

So, the only possibility is that all are even.

Let $x = 2X, y = 2Y, z = 2Z$. Then

$$4X^2 + 4Y^2 + 4Z^2 = 16XYZ$$

$\therefore \quad X^2 + Y^2 + Z^2 = 4XYZ\,.$

The same argument shows that X, Y, Z are even. The process can be continued indefinitely.

This is possible only when $x = y = z = 0\,.$

1.61: Let the number of participants from class XI be n and the score of each of them be x.

$\therefore \quad$ Sum of all the scores $= 8 + nx.$ (A)

The total number of games played is $\binom{(n+2)}{2)}$. In each game, the two players together get a score of one point.

Hence the sum of the scores must be $n + 2_{c_2}$.

Thus $\frac{(n+2)(n+1)}{1.2} = 8 + nx.$ (B)

i.e., $n^2 + 3n + 2 = 16 + 2nx.$ (C)

i.e., $n(n + 3 - 2x) = 14.$ (D) Since $2x$ is non-negative integer, we see that $n/14$. (E)

Hence $n = 7$ or $14.$ (F)

The number of participants from class XI can be either 7 or 14. (G)

Consequently x can be either 4 or 8.

1.62: Let S_{n-1} contain m terms. Then S_n contains $m + (m-1) = (2m-1)$ terms. By induction on n, it follows that there are $2^n + 1$ terms in S_n. In fact, there are $2^{100} + 1$ terms in S_{100}. Let a, b be consecutive terms in some s_n. If $a > b$ then a is the new term introduced in the formation of s'_n and it is formed from the consecutive

terms $a-b$, b in s_{n-1}. If $a<b$, then, b is the new term and the consecutive terms in s_{n-1}, forming it are $a, b-a$. Since a common divisor of a, b is also a divisor of $(a-b)$, $(b-a)$, we see successively that a is relatively prime to b. It follows that 20 appears as a new term in some s_n is one of the following ways: $1+19$, $3+17$, $7+13$, $9+11$, $11+9$, $13+7$, $17+3$, $19+1$. Hence 20 occurs 8 times in s_{100}.

1.63: Consider the numbers

$\frac{k(k+1)}{2}$ for $k=0,1,2,\ldots$ i.e., the numbers 0,1,3,6,10,15.... Any $n \geq 0$ lies between two consecutive terms in this.

i.e., $\dfrac{k(k+1)}{2} \leq n < \dfrac{(k+1)(k+2)}{2}$ for since $k \geq 0$.

Put $x = n - \dfrac{k(k+1)}{2}$.

Then $0 \leq n < \dfrac{(k+1)(k+2)}{2} = \dfrac{k(k+1)}{2}$

or $0 \leq x < k+1$, or $0 \leq x < k$.

If $y = k - x$, then

$$\begin{aligned} n &= \frac{k(k+1)}{2} + X = \frac{(x+y)(x+y+1)}{2} + x \\ &= \frac{(x+y)^2 + 3x + y}{2}. \end{aligned}$$

This proves the existence of x and y.

For Uniqueness: Let x, y be like this and k be chosen as before. If $(x+y) < k-1$, then

$$n = \frac{(x+y)(x+y+1)}{2} + x < \frac{(k-1)k}{2} + k = \frac{k(k+1)}{2}$$

which is not possible. Hence $x+y \geq k$,.

If $x+y \geq, k+1$,

then $x \geq \dfrac{(x+y)(x+y+1)}{2} \geq \dfrac{(k+1)(k+2)}{2}$, which also if not true.

Hence $x+y=k$, since k is uniquely determined by n, we see that, x and y be uniquely determined by n.

1.64: Let x, y, z be real numbers with $x+y=2$ and $xy - z^2 = 1$.

Then $4-(x-y)^2 = (x+y)^2-(x-y)^2 = 4xy = 4+4z^2$.

Thus $(x-y)^2+4z^2 = 0$.

This implies $z = 0 = (x-y)$ (A)

$\therefore$ Solution is $x=1$, $y=1$, $z=0$.

1.65: Suppose that $p(x)=7$ when $x=a, b, c$ and d (A)

then $p(x)-7=(x-a)(x-b)(x-c)(x-d)q(x)$ (B)

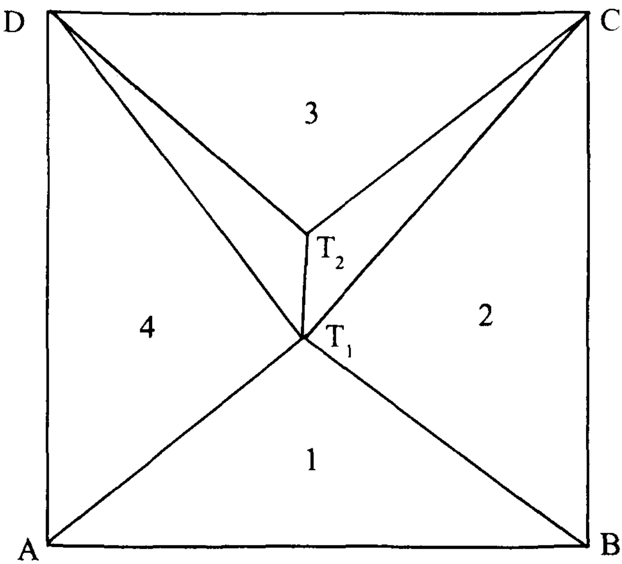
If $p(x)=14$ for some integer x, then

$7=(x-a)(x-b)(x-c)(x-d)q(x)$ for that x (C)

RHS contains of least 4 disjoint factors $(x-a)$, $(x-b)$, $(x-c)$ and $(x-d)$ but 7 has only two factors $1, 7$. Hence the proposition.

1.66: Area of square =400 square of units. Total number of points $= 2004$. Let T, be connected to A, B, C, D we get 4 triangles.

T_2 will be in the interior of one of these triangles, say T_1CD. Connect T_2 with T_1, D and C. So this triangle is divided into three triangles and the total number of



triangles is now and 6(2 more than before) The next point T_3 in whichever triangle it is, increases the total number of triangles by 2 when T_3 is joined to the vertices of the triangle where it lies. So, after all the 2000 points are placed, we get $4 + (2 + 2 + \ldots 1999)$ times $=4002$ triangles. If each triangle among these has area $\geq \frac{1}{10}$, then the total area is $(4002 \times \frac{1}{10})$ which is greater than 400 the area of the original square. This is a contradiction.

$\therefore$ There is one triangle at least with area $< 1/10$.

1.67: suppose that D is a knave. Clearly B is a knave. Since both B and D are knaves, A is a knave. Since B is a knave, what he said is not true and so C must be a knight. Thus be must have answered 'yes' to the question "Is A a knave"? and it is possible to deduce the truth about A. Thus D cannot be a knave and D is a knight.

1.68: Since 5^m and 5^k both end with 25, the last two digits of $5^m + 5^k$ are 50. A perfect square which ends with 0 should have its last two digits as 00.Thus $5^m + 5^k$ can never be the square of an integer.

1.69: $(a^2 + b^2 + (a-b)^2)^2$

$$= a^4 + b^4 + (a-b)^4 + 2a^2b^2 + 2a^2(a-b)^2 + 2b^2(a-b)^2$$
$$= a^4 + b^4 + (a-b)^4 + 2a^2b^2 + 2(a^2 + b^2)(a-b)^2$$
$$= a^4 + b^4 + (a-b)^4 + 2a^2b^2 + \{(a-b)^2 + (a+b)^2\}(a-b)^2$$
$$= a^4 + b^4 + (a-b)^4 + (a^2 - b^2)^2 + (a-b)^4 + 2a^2b^2$$
$$= 2(a^4 + b^4 + (a-b)^4).$$

Similarly $(c^2+d^2+(c-d)^2)^2 = 2(c^4+d^4+(c-d)^4)$. Since $a^2 + b^2 + (a-b)^2 = c^2 + d^2 + (c-d)^2$, the result follows.

1.70: Since 3^{1000} contains 478 digits, of

$$a \le 478 \times 9 \le 4500.$$

Thus 'a' can have not more than 4 digits. Now $b < 4 \times 9 = 36$; Since a, b, c multiples of 9, $b \in \{9, 18, 27\}$. Thus $c = 9$.

1.71: Suppose that $a_1, a_2 \dots a_9$ and $b_1, b_2, \dots b_9$ are two arrangements of the digits $1, 2, \dots 9$ and suppose that

$$\begin{array}{r} a_1a_2 \dots a_9 \\ 8 \\ \hline b_1b_2 \dots b_9 \end{array}$$

Since both the number and the answer have the same number of digits, $8a_1 < 10$. But $a_1 \ge 1$ and therefore $a_1 = 1$. Then

$$8(a_1a_2) = b_1b_2 + 1 \le 98 + 1$$

Since b_1, b_2 are different digits.

$\therefore\ 8(10 + a_2) < 99$ and hence $8a_2 \leq 19$.

But $a_2 \neq 1\ \therefore\ a_2 = 2$.

Similarly $8(a_1a_2a_3) \leq b_1b_2b_3 + 1 \leq 987 + 1$ and hence $8(120 + a_3) \leq 988$. $8a_3 \leq 28$, but $a_3 \neq 1$ and $a_3 \neq 2$.

$\therefore\ a_3 = 3$ continuing this argument we find that $a_4 = 4$, $a_5 = 5$, $a_6 = 6$, $a_7 = 7$.

We must have the only two possibilities for a_8 and a_9 namely $a_8 = 8$ and $a_9 = 9$ or $a_8 = 9$ and $a_9 = 8$. An easy verification shows that the solution is

```
1 2 3 4 5 6 7 8 9 x
                8
--------------------
9 8 7 6 5 4 3 1 2
```

1.72: Since the given sums are distinct, so must be the elements of S. Denote the elements of S by a, b, c, d, e and let $a < b < c < d < e$. It is easy to see that the smallest of the sums must be $(a + b)$ and the second smallest must be $a + c$.

Thus $a + b = 1967$ and $a + c = 1972$ (A)

Similarly the largest of the sums is $d + e$ and the second largest must be $c + e$.

$\therefore d + e = 1991$ and $c + e = 1989$ (B)

Also the ten sums are: $a + b$, $a + c$, $a + d$, $a + c$, $b + c$, $b + d$, $b + e$, $c + d$, $c + e$, $d + c$. (C)

Thus the sum of all there is

$$4(a+b+c+d+e) = 19788. \tag{D}$$

$$\therefore \quad a+b+c+d+e = 4947 \tag{E}$$

We have now five independent equations involving the five unknowns a, b, c, d, e and solving these we get

$$S = \{983, 984, 989, 991, 1000\}$$

1.73: $2mn - 5m + n = 55 \Rightarrow (2m+1)(2n-5) = 105$ (A)

Now 105 has ± 1, ± 3, ± 5, ± 7, ± 15 ,± 21, ± 35, ± 105 as divisors. These give the following 16 pairs of (m, n) satisfying the given equation. $(0, 55)$, $(1, 20)$, $(2, 13)$, $(3, 10)$, $(7, 6)$, $(10, 5)$, $(17, 4)$, $(52, 3)$, $(-53, 2)$, $(-18, 1)$, $(-11, 0)$, $(-8, -1)$, $(-4, -5)$, $(-3, -8)$, $(-2, -15)$, $(-1, -50)$.

1.74: From 1 to 100, there are 11 zeroes. From 101 to 900, there are $8 \times 20 = 160$ zeroes. From 901 to 1000, there are 21 zeroes. $\therefore$ Total number of zeroes $= 11+160+21 = 192$.

1.75: Then numbers with the given property are 1000000001, 1000000010, 1000000100, ... 1100000000 and 2000000000.

$\therefore$ The required number is 10.

1.76: Let x be the number of marbles found in the bag in the beginning. Then $\frac{x}{2}$ must be an integer if x is even. Also

$$\left[\left(\frac{x}{2}+1\right) + \frac{1}{3}\left(\frac{x}{2}-1\right)\right] = \frac{2x+2}{3}$$

must be an integer. Hence $2x+2$ is a multiple of 3.

As $2x+2$ is also even, $(2x+2)$ is a multiple of 6.

$\therefore$ $2x$ is a multiple of 6 plus 4. or x is a multiple of 3 plus 2. As x is even, it should also be a multiple of 2. Hence x is a multiple of 6 plus 2.

1.77: $1000^{20} = 1000\ldots00$ (one followed by 60 zeroes).
$\therefore$ $1000^{20} - 20 = 99\ldots980$ (58 nines followed by 80).

$\therefore$ Sum of the digits $= (9 \times 58) + 8 = 530$.

1.78: The required numbers are

1111	1221	1331	1441	
1122	1232	1342		
1133	1243			(10)
1144				
2112	2211	2321	2431	
2123	2222	2332	2442	
2134	2233	2343		
	2244			(12)
3113	3212	3311	3421	
3124	3223	3322	3432	
	3234	3344	3443	(11)
4114	4213	4323	4411	
	4224	4334	4422	
			4433	
			4444	(9)

The total of such number is 42.

1.79: (a) If $x = 0$, then the given equation does not hold. If $x < 0, |x| = -x$ and the equation is

$$x^2 - 2x + 1 = 0 \text{ or } (x-1)^2 = 0.$$

Hence $x = 1 > 0$ a contradiction.

If $x > 0, |x| = x$ and the equation is $x^2 + 2x + 1 = 0$ or $(x+1)^2 = 0$ or $x = -1$ $\therefore$ $x = -1 < 0$, again a contradiction.

When x is real, it has to be positive, negative or zero. As all these three cases lead to a contradiction, x is not real. The equation $x^2 + 2|x| + 1 = 0$ has no real roots.

1.79: (b)

$$\begin{aligned} \left[\tfrac{1}{4} + \tfrac{x}{50}\right] &= 0 \quad \text{for } x \leq 37 \text{ and} \\ \left[\tfrac{1}{4} + \tfrac{x}{50}\right] &= 1 \quad \text{for } x = 38, 39, 40 \end{aligned}$$

$$\sum_{x=1}^{40}\left[\frac{1}{4} + \frac{x}{50}\right] = 3$$

1.80: Let $x^{2002} - 2001 = Q.x^{91} + R$ so that Q is the quotient and R is the remainder required.

$$\begin{aligned} 2002 &= 91 \times 22 \\ \frac{91 \times 22 - 2001}{x^{91}} &= x^{91\times 21} - \frac{2001}{x^{91}} \end{aligned}$$

$\therefore$ $Q = x^{91\times 21}$ and $R = -2001$.

1.81: Let A take x cards and give y cards to B initially. Then obviously $x + y < 52$. (A)

It is given that $y > x$. (B)

If A transfers k of his cards to B,

then $y + k = 4(x - k)$ (C)

and $y - k = 3(x + k)$ (D)

i.e., $4x - y = 5k$ and $3x - y = -4k$ (E)

$\therefore$ $x = 9k$ and $y = 31k$ (since $y > x$).

As $x + y < 52, 40k < 52$ and k is an integer.

Thus $k = 1$. $\therefore$ $x = 9$ and $y = 31$.

1.82: If $x^2 + 2x + 5$ is a factor of $x^4 + px^2 + q$, then the other factor is of the from $(x^2 + ax + b)$.

$\therefore$ $(x^2 + 2x + 5)(x^2 + ax + b) = x^4 + px^2 + q$.

i.e., $x^4 + (2 + a)x^3 + (5 + b + 2a)x^2 + (5a + 2b)x + 5b = x^4 + px^2 + q$.

$\therefore$ $2 + a = 0 \Rightarrow a = -2$

$5 + b + 2a = p \rightarrow 5 + b = p + 4$ and
$5a + 2b = 0 \rightarrow 2b = 10$. $\therefore b = 5$

$\therefore$ $p = 6$ and $q = 25$ as $q = 5b$.

$\therefore$ The other factor is $x^2 - 2x + 5$.

1.83: Let x students wear spectacles and y students bring lunch. Students who either wear spectacles or bring lunch or both $= x + y - 60$ (A)

Hence the number who do not wear spectacles and do not bring lunch $= 500 - (x + y - 60) = 560 - x - y$ (B)

This number is also equal to $\frac{2}{3}(500 - x)$ or $\frac{3}{4}(500 - y)$.
$\therefore$ $\frac{2}{3}(500 - x) = \frac{3}{4}(500 - y) = 560 - x - y$ (C)

i.e., $4000 - 8x = 4500 - 9y = 6720 - 12 - 12y$ (D)

$4x - 12y = 2720$ and $12x + 2y = 2220$. (E)

Solving $y = 180$ and $x = 140$. (F)

$\therefore$ Required number is $\frac{2}{3}(500 - 140) = 240$. (G)

1.84: Let M, W, C be the number of men, women and children respectively. Then,

$$\begin{aligned} M + W + c &= 100 \\ 50M + 10W + \frac{c}{2} &= 1000 \\ 100M + 20W + c &= 2000 \\ \therefore \quad 99M + 19W &= 1900. \end{aligned}$$

For integral solution, $W = 1$ gives $x = 19$.
$$\therefore \quad c = 80.$$

1.85: Let P's age be x years and Q's age be y years.

Then $x^2 - y = 158$ and $y^2 - x = 108$.

The first square number greater than 158 being 169, $x = 13$ and $y = 11$.

1.86: $P(x + \frac{3}{2}) = P(x)$ implies
$$P(8) = P(5 + \tfrac{3}{2} + \tfrac{3}{2}) = P(5 + \tfrac{3}{2}) = P(5) = 2006.$$

1.87: Now $\left(a + \frac{1}{b}\right)\left(b + \frac{1}{c}\right)\left(c + \frac{1}{a}\right)$

$$\begin{aligned} &= abc + (a + b + c) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{1}{abc} \\ &= abc + \frac{1}{abc} + \left(\left(a + \frac{1}{a}\right) + \left(b + \frac{1}{b}\right) + \left(c + \frac{1}{c}\right)\right) \\ &= abc + \frac{1}{abc} + \left(\frac{7}{3} + 4 + 1\right) \\ &= abc + \frac{1}{abc} + \frac{22}{3}. \end{aligned}$$

Now $\left(a + \frac{1}{b}\right)\left(b + \frac{1}{c}\right)\left(c + \frac{1}{a}\right) = \frac{7}{3} \times 4 \times 1 = 28/3$.

$$\therefore \quad abc + \frac{1}{abc} = 2$$

$$\therefore \quad (abc - 1)^2 = 0 \quad \therefore \quad abc = 1\,.$$

1.88: LHS is divisible by 9. $\therefore$ RHS is divisible by 9 (and also $0 < a \leq 9$)

$\therefore \quad 4 + 9 + 2 + a + 0 + 4$ i.e., $19 + a$ is divisible by 9.

$\therefore \quad a = 8\,.$

1.89: There cannot be a pythagorean triple where each of the numbers is odd; for, if a, b are odd then $a^2 + b^2$ is even and c cannot be odd. Number of pythagorean triples (a, b, c) where $a^2 + b^2 = c^2$ and each integer a, b, c being odd is zero.

1.90: $1, 2, 7, 286, 182, 11, 13, 154, 1001, 2002$ are the divisors of 2002.

The largest $I + M + 0 = 1 + 2 + 1001 = 1004\,.$

1.91: In 1994, if age of Ram is x, then his grandfather's age is $2x$. So their birth took place in $(994 - x)$ and $(1994 - 2x)$ respectively. Thus $3844 = 3988 - 3x$ so that $x = 48$. Hence in 2001, the age of Ram is $x + 7 = 55$.

1.92: Sum of the squares $= 0 \rightarrow$ each term $= 0$

$\therefore a = 3/2\,,\ b = 5/4\,,\ c = 7/6$

$\therefore a + b + c = 47/12\,,\ abc = 105/48\,.$

$\therefore \quad abc < (a + b + c).$

1.93: $N = 3000003 + 10000a + 100b = 6 + 3a - 4b +$ some multiple of 13.

$\therefore$ N will be divisible by 13 if $(3a - 4b + 6)$ is exactly divisible by $13 (0 \le a \le 9; 0 \le b \le 9)$.

If $a = 3, b$ can be 7;

If $a = 2, b$ can be 3;

If $a = 0, b$ can be 8.

But if $a = 1$, b is not an integer. Thus 'a' can take the values $3, 2, 0$ but not 1.

1.94: Taking $x = 1000$, the sum is

$$\begin{aligned}(1 + 7 + 13 + 19 + \ldots + 6001) &- (4 + 10 + 16 + \ldots + 5998)\\ &= \frac{1001}{2}(6001 + 1) - \frac{1000}{2}(5598 + 4)\\ &= 3001.\end{aligned}$$

1.95: There are 20 gaps and each gap corresponds to a handshake. Thus on the whole, the number of hand shakes is 20.

1.96: Minute hand should be between 4 and 5. Hour hand should be between 5 and 6. Let the hand be x minutes-divisions beyond 5. Therefore actual time $= 12x$ minutes past 5. But it was taken to be $(x + 25)$ minutes past 4.

$\therefore$ Difference $= (60 + 12x) - (x + 25) = 11x + 35$. Thus $11x + 35 = 57$ $\therefore$ $x = 2$ $\therefore$ Correct time is 24 minutes past 5.

1.97: The given equation is

$(2x + 3)(x^{2000} + x^{1998} + \ldots + x^2 + 1) = 0$.

Can not is $-\frac{3}{2}$ which is real.

The other roots are given by $x^{2000}+x^{1998}+\ldots+x^2+1=0$. But all the roots of this equation are imaginary since all the terms are positive and for no real x, their sum can be zero. $\therefore$ The number of real roots is one only.

1.98: Equation with roots 8 and 2 is

$(x-8)(x-2)=0$ i.e., $x^2-10x+16=0$ (A)

Equation with roots $-9,-1$ is $x^2+10x+9=0$. (B)

The constant term of (A) is wrong; x term of (B) is wrong.

$\therefore$ The correct equation is $x^2-10x+9=0$.

1.99: Let Rs. m be the cost of 1 mathematics book and Rs. s be the cost of 1 science book

$\therefore \quad 9m+16s=220$ (A)

$m=25-\dfrac{5+16s}{9}$ (B)

As m is a natural number less than 25, $(5+16s)$ is a multiple of 9. (C)

$\therefore \quad s=7$ (D)

$m=25-13=12$. (E)

Thus the cost of each mathematics book is Rs.12.

2.00: Since a good number n is the sum of two consecutive integers, it must be odd, and also since it is the sum of three consecutive integers, it must be divisible by 3. So, for some integers, m and k. $m=3k=2m+1$.

Now $n=m+(m+1)$ and $n=(k-1)+k+(k+1)$. Therefore every odd multiple of 3 (bigger than 3) is "good".

i.e., a number is 'good' if it is an odd multiple of 3 (< 3).

(i) Thus 2001 is 'good' but 3001 is not 'good'.

(ii) The product of any two odd multiples of 3 is again an odd multiple of 3 and hence the product of two good numbers is always good.

(iii) If the product of two natural numbers x and y is good, then xy is an odd multiple of 3. This implies that at least one of them is an odd multiple of 3. Thus, if the product of two good numbers is good, then, at least one of them, is good.

2.01: $x^2 + ax + b^2 = 0$ and $x^2 + bx + a^2 = 0$ have a common solution α implies.

$$\alpha^2 + \alpha + b^2 = \alpha^2 + b\alpha + a^2$$

$$\therefore \quad \alpha(a-b) = a^2 - b^2 \Rightarrow a = b \text{ or } \alpha = a + b. \qquad \text{(A)}$$

So either $a = b$ or the common root ie $\alpha = (a+b)$

Case 1: Let $a = b$. In this case, the two equations are identical and they have real roots if $a^2 - 4b^2 = -3a^2 = 0$ which is possible if $a = b = 0$.

Case 2: $\alpha = (a+b)$ is the common root.

In this case, we have

$$(a+b)^2 + a(a+b) + b^2 = 2a^2 + 3ab + 2b^2 = 0$$

Since 'a' is given to be real number, the discriminant of this quadratic in 'a' must be positive.

Hence $9b^2 - 16b^2 = -7b^2 \geq 0$. This can happen only if

$b = 0$. $\therefore$ The answer to the problem is $a = b$ or for real roots, $a = b = 0$.

2.02: Among any six students, there are two who have the same age. We can divide the whole group into at most five different classes such that each class contains students with the same age. Since $[\frac{281}{5}] = 56$, there is at least one class having at least 57 students having the sam age. In this group of 57, since there are only 7 countries, there must at least be 9 people from the same country. In this nine, at least five must be of the same gender. Hence the result.

2.03: Let the lengths of the ten segments be arranged as
$1 \le a_1 \le a_2 \le a_3 \ldots \le a_{10} < 55.$ (A)

Assume that triangle can be constructed. Then,

$$a_3 \ge a_1 + a_2 > 2$$
$$a_4 \ge a_2 + a_3 > 1 + 2 = 3$$
$$a_5 \ge a_3 + a_4 > 2 + 3 = 5a_6 \qquad \ge a_4 + a_5 > 3 + 5 = 8$$
$$a_7 \ge a_5 + a_6 > 5 + 8 = 13$$
$$a_8 \ge a_6 + a_7 > 8 + 13 = 21$$
$$a_9 \ge a_7 + a_8 > 13 + 21 = 34$$
$$a_{10} \ge a_8 + a_9 > 21 + 34 = 55$$

Hence $a_{10} > 55$ which is a contradiction to our assumption. Hence we can select three sides of a triangle with the given segments.

2.04: Let us list the positive integral values of x satisfying the condition $[\frac{x}{99}] = [\frac{x}{101}]$ and count the number of such values of x.

Positive integral values of x satisfying the given condition	The value of $[\frac{x}{99}] = [\frac{x}{101}]$	Total number of values of x
1 to 98	0	98
101 to 197	1	97
202 to 296	2	95
303 to 395	3	93
404 to 494	4	91
4848 to 4850	48	3
4949 only	49	1

Hence total number of such values of

$$x = 98+97+95+\ldots 1 = (1+3+5+\ldots+97)+98 = 2499.$$

For other values of x, $\left[\frac{x}{99}\right] \neq \left[\frac{x}{101}\right]$

2.05: The expression $(2.4.6\ldots 1000 \text{ factors})$

$$-(1.3.5\ldots 1000 \text{ factors}) = (2.4.6\ldots 2000)-(1.3.5\ldots 1999).$$

Now $2001 = 3 \times 23 \times 29$ and $3, 23, 29$ are co-primes two by two. As $6, 46, 58$ are divisible by $3, 23, 29$ respectively and these occur as factors in $2 \cdot 4 \cdot 6 \ldots 2000$, the product $2 \cdot 4 \cdot 6 \ldots 2000$ is divisible by 2001. Also, as $3, 23, 29$ occur as factors in the product $1 \cdot 3 \cdot 5 \ldots 1999$, this product $1 \cdot 3 \cdot 5 \ldots 1999$ is also divisible by 2001.

Hence their difference $(2 \cdot 4 \cdot 6 \ldots 2000) - (1 \cdot 3 \cdot 5 \ldots 1999)$ is divisible by 2001.

2.06: We have

$$X = a^3+b^3+c^3-3abc = \frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$$

since we are interested in the least positive value of X, a, b, c are not equal.

Therefore, the integers should satisfy

$$a + b + c \geq +1 + 1 + 2 = 6.$$

$(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0 + 1 + 1 + 2.$

This means that $X \geq 4$. If $a = 1, b = 1, c = 2$, we get $X = 4$.

Hence the least positive value of X is 4 and it is obtained when $(a, b, c) \in \{(1,1,2),(1,2,1),(2,1,1)\}$.

2.07: Assume that there exists such a set a, b, c forming an A.P. in some order. We may assume that $a < b < c$. Then, there are only two possibilities.

Case(i):

$a+b-c < a < c+a-b < b < c < b+c-a < a+b+c.$

In this case, if d is the C.D of the progression, then $c = a+b+c-2d, b = a+b+c-3d, a = a+b+c-5d$. Adding, $a+b+c = 3(a+b+c) - 10d$.

$$\Rightarrow a+b+c = 5d \Rightarrow a = 0.$$

This contradicts are assumption that $a > 0$.

Case 2:

$a+b-c < a < c+a-b < b < c < b+c-a < a+b+c.$

In this case,we have $c = a+b+c-2d$, $b = a+b+c-4d$, $a = a+b+c-5d$.

Adding, $a+b+c = 3(a+b+c) - 11d$

$$\Rightarrow a+b+c = \frac{11d}{2}$$

$$\therefore \quad a = \frac{d}{2}; b = \frac{3d}{2}; c = \frac{7d}{2}.$$

Also we have,

$$a+b-c = (a+b+c) - 6d.$$

$$= \frac{11}{2}d - bd = -d/2$$

$$\therefore \quad a+b-c = (1+3-7)d/2 = -\frac{3d}{2}.$$

This implies that $d = 0$ which again leads to a contradiction. Thus there is no such set a, b, c of positive numbers.

2.08: For any integer n, we always have $n^2 \equiv 0, 1, 2 \pmod 8$ so the different possibilities for $a^2 + b^2$ modulo 8 are $0+0=0$, $0+1=1$, $1+1=2$, $1+4=5$, $0+4=4$, $4+4=0$.

But $a^2+b^2-8c=6$ implies $a^2+b^2 = 6(\text{mod}8)$ which is impossible as per the above observation. Hence there are no integers a, b, c for which $a^2+b^2-8c=6$.

2.09: Suppose the roots x_1, x_2, x_3 of $ax^3+bx^2+cx+d=0$ are all rational numbers. Then the transformation $y = ax$ transforms the cubic into

$$a\left(\tfrac{y}{a}\right)^3 + b\left(\tfrac{y}{a}\right)^2 + c\left(\tfrac{y}{a}\right) + d = 0$$

i.e., $y^3 + by^2 + acy + a^2d = 9.$ (A)

The roots of this equation are

$$y_1 = ax; y_2 = ax_2; y_3 = ax_3 \quad \text{(B)}$$

Now y_1, y_2, y_3 are all rational numbers. (C)

As the coefficients of equation (A) are all integers.

We must have y_1, y_2, y_3 as integers. (D)

Further, $y_1 + y_2 + y_3 = -b$ (E)

$y_1y_2 + y_2y_3 + y_3y_1 = ac$ (F)

$y_1y_2y_3 = -a^2d$ (G)

Now, in view of equation $(G,)y_1, y_2, y_3$ must be divisors of a^2d.

Since it is given that 'ad' is an odd integer.

y_1, y_2, y_3 are all odd integers. (H)

From equations (E) and (F), it is clear that b and ac must be odd. (I)

This means a, b, d are all odd integers. (J)

Therefore equation (F) implies that 'c' is an odd integer. (K)

Now b and c are odd contradicts our hypothesis that 'bc' is even. (L)

So the given equation has at least one irrational root. (M)

2.10: $f(1) + 2f(2) + 3f(3) + \ldots + nf(n) = n(n+1)f(n)$

$f(1)+2f(2)+3f(3)+\ldots+(n-1)f(n-1) = (n-1)nf(n-1)$

for $n \geq 3$, then subtracting the second equation from the first, $n^2 f(n) = n(n-1)f(n-1)$ or $(n-1)f(n-1) = nf(n)$, for $n \geq 3$.

$\therefore \quad 2f(2) = 3f(3) = 4f(4) = \ldots nf(n)$ for $x \geq 2$.

From b with $n = 2004$, we get $f(1) + 2f(2) + \ldots +2003f(2003) = (2004)^2 f(2004)$

or $1 + (2002) \times (2004)f(2004) = (2004)^2 f(2004)$.

$$\therefore \quad f(2004) = \frac{1}{(2004)(2004-2002)} = \frac{1}{4008}$$

2.11: Let $a_1, a_2, a_3 \ldots a_x$, be the sequence real numbers. We write the successive 7-sums in separate rows. If we continue writing till the first 11 such 7-sums we get 11 rows of 7 sums which in turn gives 7 columns of 11 sums.Let us construct a table.

$$\begin{array}{rcl}
a_1 + a_2 + a_3 + \ldots + a_7 & < & 0 \\
a_2 + a_3 + a_4 + \ldots + a_8 & < & 0 \\
a_3 + a_4 + a_5 + \ldots + a_9 & < & 0 \\
\ldots & < & 0 \\
\ldots & < & 0 \\
\ldots & < & 0 \\
a_{11} + a_{12} + a_{13} + \ldots + a_{17} & < & 0
\end{array}$$

In the above table, each row sum is negative and so the sum of all the terms is negative. On the other hand, each column sum is positive and so the sum of all the terms is positive. This leads to a contradiction. Therefore at best we can have 10 such, 7-sum rows. This means that the

sequence can have at most 16 terms. By trial and error method we construct the sequence of 16 terms. 5, 5, -13, 5, 5, 5, -13, 5, 5, -13, 5, 5, 5, -13, 5, 5 satisfying the given requirements.

2.12: From $\bar{3a} \times 2 = 70$

```
      3a
  x   b2
   -----
      70
    140
  ------
    1470
```

We get $a = 5$. Now $\bar{3a} \times b = 140$ or $35 \times b = 140$. $\therefore \quad b = 4$. $\therefore \quad a + b = 5 + 4 = 9$.

2.13: There are 12 months in a year. If at all, all the invitees to the party are born different months, it can be a maximum of 12. The 13^{th} invitee if any, should be born a month which belongs to any month among these 12 only. So the number of people to be invited for the party cannot exceed 12.

2.14: $x = 9 + 4\sqrt{5}$;

$$y = \frac{1}{x} = \frac{1}{9 + 4\sqrt{5}} = \frac{9 - 4\sqrt{5}}{(9 + 4\sqrt{5})(9 - 4\sqrt{5})}$$

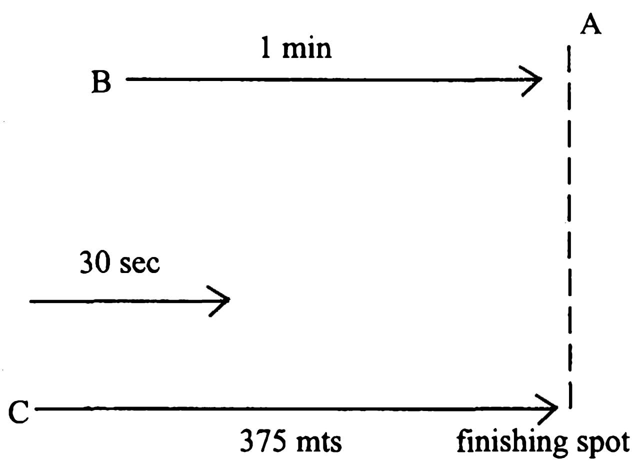
$$\therefore \quad \frac{1}{x^2} + \frac{1}{y^2} = x^2 + \frac{1}{x^2} = (x + \frac{1}{x})^2 - 2$$

$$= 18^2 - 2 = 322.$$

2.15: A beats C by $(60+30) = 90$ seconds

Time taken for $375m = 90$.

Time taken for $1000m = \dfrac{90}{375} \times 1000 = 240$ secs. Assuming



that B, C are traveling at the same speed, the distance between B and C would be the same throughout (After A finishes the race). From the picture,

```
      Time takes          Distance travelled
      in seconds          or can be travelled
                          by C in meters
--------------------------------------------
              90                  375
            + 90                + 375
            ----                 ----
             180                  750
div 3 =>      60     div 3 =>     250
            ----                 ----
Adding       240                 1000 m
            ----                 ----
```

C could run 1 km in 240 seconds.

2.16: Given $a+b+c=0 \to b=-(a+c)$.

$$\therefore \quad b^2=(a+c)^2=a^2+c^2+2ac$$

$$\therefore \quad \frac{b^2+c^2+a^2}{b^2-ca}=\frac{2(a^2+c^2+ac)}{(a^2+c^2+ac)}$$

$$=2.$$

2.17: $|\ldots x$ years$\ldots|\ldots x$ years$|$

Let Ram birth year $=x$.

Grandpa $=y$; $x+y=3854$ (1)

$$\begin{aligned} \text{In2002,} \qquad 2(2002-x) &= 2002-y \\ 2x-y &= 2002 \qquad (2) \\ \therefore \quad x &= 1952 \end{aligned}$$

Adding (1) and (2), $3(y+x)=5856$. $\therefore \quad y+x=1952$.

$\therefore$ This age of year 2003 will be 2003-1952=51.

2.18: 18 and 21 are multiples of 3.

$\therefore$ the natural numbers multiplied by 18,21 are also multiples of 3.

Then sum should also be a multiple of 3; now 2004 satisfies this condition while 2005 or 2006 do not satisfy.

$\therefore$ 2004 could be the sum of the products as wanted.

2.19: The range of score marks will be from o to 100 (in whole numbers) Number of students who can score different marks is a maximum 101.

But there are 120 students. The remaining 19 students must score marks that are already scored by any one of the 101 students.

Since no three students score the same mark only 2 students could have scored the same mark.

The smallest possible number of students in pairs who are awarded the same rank (or scored the same rank) is 19.

2.20: Volume of 27 metal balls = volume of lift sphere

$$= 27 \times \frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3 \Rightarrow R = 3R. \qquad \text{(A)}$$

$$\therefore \quad \frac{\text{surface area of the big sphere}}{\text{surface area of the ball}} = \frac{4\pi R^2}{4\pi r^2} = \frac{R^2}{r^2}$$

$$\text{Ratio } = 9 : 1 = \frac{9r^2}{r^2} = \frac{9}{1}$$

2.21: 36 can be expressed as the product of any number of 1's two 2's and two 3's.

$$\text{i.e., } 36 = \underbrace{1 \times 1 \times 1 \times \ldots \times 1}_{\text{any number of times}} \times 2 \times 2 \times 3 \times 3$$

The factorization of 36 should be regrouped into the product of 3 terms such that the sum of 3 terms is an odd prime.

The three terms can be (i) all odd (ii), 1odd, 2 even. But all the terms cannot be odd because of the presence of s and $2's$.

$\therefore$ The three terms should be 1 odd, 2 even. the possibilities are 1,6,6 or 2,2,9 or 1,2,18. But 1+2+18=21 is

not a prime and there is only one younger sister who likes ice-cream.

$\therefore$ The ages of sisters are 1,6,6 respectively and the product the elder sister ages is $6 \times 6 = 36$.

2.22: Let $\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \ldots = \frac{x_{15}}{x_{16}} = \frac{1}{k}$.

Given

$x_1 + x_2 + x_3 + x_4 = 20 \rightarrow x_1[1 + k + k^2 + k^3] = 20$

$x_5 + x_6 + x_7 + x_8 = 320 \Rightarrow x_1[k^4 + k^5 + k^6 + k^7] = 320$

(or) $x_1 k^4[1 + k + k^2 + k^3] = 320$

$\Rightarrow 20 \times k^4 = 320 \Rightarrow k = 2$

$x_{13} + x_{14} + x_{15} + x_{16}$

$= x_1[k^2 + k^{13} + k^{14} + k^{15}]$

$= x_1[1 + k + k^2 + k^3] \times k^{12}$

$= 20 \times 2^{12} = 81920.$

2.23: $kx + y = 4;\ x + ky = 5$

$\Rightarrow (k+1)[x+y] = 9$

or $(k+1)2 = 9 \Rightarrow = 35$ (A)

The number of values of k for which the system of equations has at least one solution is 1.

2.24: $S_{2002} = 1 - 2 + 3 - 4 + 5 - 6 + \ldots + 2001 - 2002$

$$= \underbrace{(-1) + (-1) + (-1) + \ldots + (-1)}_{\text{1001 times}}$$

$\Rightarrow S_{2002} = -1001$ (A)

Similarly $2_{2004} = -1002$ (B)

But $S_{2003} = -1001 + 2003 = 1002$ (C)

Therefore,

$S_{2002} - S_{2003} + S_{2004} = -1001 - 1002 - 1002 = -3005$ (D)

2.25: Since no repetition of digits is allowed, the five digit numbers can be formed such that each digit from 1,2,3,4,5 represents every place value (ten thousand's, thousand's, hundred's, ten's and units) 24 times.

This is because, for a digit being fixed in one place value, the other digits can occupy the remaining 4 places in $4 \times 3 \times 2 \times 1$ ways. The sum of all five digit numbers

$$\begin{aligned} &= 24 \times (1+2+3+4+5) \times \\ &\quad \times [10000 + 1000 + 100 + 10 + 1] \\ &= 24 \times 15 \times 11111 \\ &= 360 \times 11111 = 3999960 \\ \text{i.e., } x &= 3999960 \end{aligned}$$

2.26: Let $m = (x+y)$ and $n = x/y$. $\Rightarrow m + n = \frac{1}{2}$ and $mn = -\frac{1}{2}$ From (given data) (A)

$$(m-n)^2 = (m+n)^2 - 4mn$$
$$= 1/4 + 2 = \frac{9}{4}$$

$m - n = \pm 3/2$

From (1) and (2) $\begin{matrix} m = 1 & \text{and} & n = -\frac{1}{2} \\ m = -\frac{1}{2} & \text{and} & n = 1 \end{matrix}$

We get two systems of equations

$$x + y = 1$$
$$x/y = -\frac{1}{2}$$
$$\text{and } x + y = -\frac{1}{2}$$
$$x/y = 1$$

Solving we get (x, y) as $(-1, 2)$ and $(-1/4, -1/4)$.

$\therefore$ The number of (x, y) pairs is 2.

2.27: Given $(x - 2)$ divides $(2x + y)$ or

$$\frac{2x + y}{x - 2} = \frac{2x - 4 + y + 4}{x - 2} = 2 + \frac{y + 4}{x - 2}$$

$\Rightarrow (x - 2)$ divides $y + 4$ for only four integral values of m clearly prime numbers has 4 factors which are integers.

$\therefore$ $(y + 4)$ should be a prime for $1 \leq y \leq 20$. The primes from 5 to 24 are $5, 7, 11, 13, 17, 19, 23$.

$\therefore$ The number of such possibilities is 7.

2.28: Let $\frac{n}{2003} = d$ and let the last 6 digits of d be $ABCDEF$. Then, we have

$$d = \ldots ABCDEF,$$
$$2003 \times d = m = 55555\ldots555$$

It is clear that f must be 5 since $3 \times F$ ends in 5. This means $(E \times 3) + 1$ ends in 5. This forces E be 8. Thus proceeding in this way, we get $A = 3$, $B = 9$, $C = 5$, $D = 1$, $E = 8$, $F = 5$. $\therefore$ The last six digits of d are $\ldots 395185$.

2.29: $x^3 + 11^3 = y^3$ (i)

$\therefore \quad 11^3 = y^3 - x^3 = (y-x)(y^2 + x^2 + yx)$

Hence $y - x = 1$ or $y - x = 11$ or $y - x = 11^2$ or $y - x = 11^3$.

Case1: Let $y - x = 1$ i.e., $y = x + 1$

$$\begin{aligned}\therefore \quad 11^3 = 1331 = 2(\mod 3) &= y^2 + x^2 + yx \\ &= (x+1)^2 + x^2 + x(x+1) \\ &= 3x^2 + 3x + 1 = 1(\mod 3)\end{aligned}$$

which says $2 = 1(\mod 3)$.

$\therefore$ Case 1 is impossible.

Case 2: Let $y = x + 11$

$$\begin{aligned}\therefore \quad 11^3 = 1331 &= (y-x)(y^2 + x^2 + yx) \\ &= 11(x+11)^2 + x^2 + x(x+11) \\ &= 11(3x^2 + 33x + 121)\end{aligned}$$

$\therefore \quad 3x^2 + 33x + 121 = 121$. $\therefore \quad x = 0, -11$.

Case 3: Let $y = x + 121$. In this case we get

$$\begin{aligned}11^3 &= 1331 = (y-x)(y^2 + x^2 + yx) \\ &= 121[(x+121)^2 + x^2 + x(x+121)] \\ &= 121[3x^2 + 363x + 121^2] \\ \Rightarrow \quad & 11 = 3x^2 + 363x + 121^2\end{aligned}$$

Here discriminant < 0. Hence there is no real solution in this case.

Case:4 Let $y = x + 11^3$.

$\therefore$ In this case also, similar analysis as in case 3 will thus that there is *no real* solution.

Thus the only solution are $(-11, 0)$ and $(0, 11)$.

2.30: Let $y(x) = ax^2 + bx + c$ and $y(x_1) = y_1, y(x) = y_2$ and $y(x_3) = y_3$ where $x_1, x_2, x_3, y_1, y_2, y_3$ are rational numbers. then we have

$$\begin{aligned} y_1 &= ax_1^2 + bx_1 + c \\ y_2 &= ax_2^2 + bx_2 + c \\ y_3 &= ax_3^2 + bx_3 + c \end{aligned}$$

This gives
$$\begin{aligned} \frac{y_1 - y_2}{x_1 - x_2} &= a(x_1 + x_2) + b \\ \frac{y_1 - y_3}{x_1 - x_3} &= a(x_1 + x_3) + b \\ \therefore \quad \frac{y_1 - y_2}{x_1 - x_2} - \frac{y_1 - y_3}{x_1 - x_3} &= a(x_2 - x_3) \\ \therefore \quad a &= \frac{1}{x_2 - x_3}\left[\frac{y_1 - y_2}{x_1 - x_2} - \frac{y_1 - y_3}{x_1 - x_3}\right] \end{aligned}$$

This proves that a is a rational number which in turn implies that b and c are rational numbers.

2.31: Assume that two elements of A differ by 3,6,9,12,18 or 21. Therefore any two elements of A differ by either non-multiple of 3 or by $24, 27, 30\ldots$ Define

$$B_1 = \{x | 1 \leq x \leq 106, x = 0(\text{mod}3)\}$$

$$B_2 = \{x | 1 \leq x \leq 106, x = 1(\text{mod}3)\}$$

$$B_3 = \{x | 1 \leq x \leq 106, x = 2(\text{mod}3)\}$$

Then we have

$$|B_1| = 35$$

$$|B_2| = 36$$

$$|B_3| = 35$$

If the difference of two numbers in A is a multiple of 3, then they both must belong to the same B and their minimum difference is 24. Hence A can contain at most 5 elements from each B_i. This accounts for 15 elements of A. Now, the 16th element differs from one of the earlier chosen element of A by a multiple of 3. But in our hypothesis, we do not allow a difference of $6, 9, 12, 15, 18, 21$.

$\therefore$ The difference must only be 3.

Hence the result.

2.32: All two-digit numbers that end with o or 5 do not contribute (since a product of these will always end with 0,5, not 6.)

11 multiplied by $16, 26, \ldots 96$ account for 9 numbers.

12 multiplied by $13, 23, \ldots 93$ account for 9 numbers.

12 multiplied by $18, 28, \ldots 98$ account for 9 numbers.

Similarly $13, 14, 16, 17, 18$ and 19-all these account for $6 \times 9 = 54$ numbers.

In total, where one of the numbers lies between 10 and 20 we could locate a total of $10 \times 9 = 90$ pairs.

This can be repeated for pairs with one of the numbers lying between 20 and 30, 30 and 40 ... and 90 and 100.

thus total number of pairs will be $9 \times 90 = 810$.

However, since we have considered all pairs, the number of unordered such pairs will be half of 810. $\therefore$ Number of distinct pairs $= \frac{810}{2} = 405$.

2.22: In the unit's place, each of the digits will occur

$$\frac{256}{4} = 64 \text{ times .}$$

In the ten's place each of the digits will occur $\frac{256}{4} = 64$ times.

In the hundred's place, each of the digits will occur $\frac{256}{4} = 64$ times.

In the thousand's place each of the digits will occur $\frac{256}{4} = 64$ times.

$\therefore$ Total $= 64(1+2+3+4) + (64)(10)(1+2+3+4) + (64)(100)(1+2+3+4) + (64)(1000)(1+2+3+4) = 711040$.

2.34: Range (x, y) and number of numbers satisfying the given conditions are given as a triple (x, y, z).

For example, $(12, 19; 8)$ represents the eight numbers $12, 13, 14, 15, 16, 17, 18, 19$ in the range $12 - 19$.

$$(23, 29; 7) \rightarrow 23, 24, 25, 26, 27, 28, 29$$

$$(34, 39; 6) \rightarrow 34, 35, 36, 37, 38, 39$$

$$(45, 49; 5) \rightarrow 45, 46, 47, 48, 49,$$

$$(56, 59; 4) \longrightarrow 56, 57, 58, 59$$
$$(67, 69; 3) \longrightarrow 67, 68, 69$$
$$(78, 79; 2) \longrightarrow 78, 79$$
$$(89; 1) \longrightarrow 89$$

So that in the range $12 - 89$, there are $8+7+6+5+4+3+2+1 = 36$ numbers. i.e., $(12, 89, 36)$. (A)

Similar three digit numbers are

$(123, 127; 7), (134, 139; 6), (145, 149; 5)$

$(156, 159; 4), (167, 169; 3), (178, 179; 2), (189; 1)$. (B)

$(234, 239; 6), (245, 249; 5), (256, 259; 4)$

$(267, 269; 3), (278, 279; 2), (289; 1)$. (C)

Upto 289 there are 36+(7+6+...+1)+(6+5+...+1) = 85 numbers. (D)

i.e., (12, 289; 85). Proceeding thus, (345, 349; 5); (356, 359; 4) (367, 369; 3); (378, 379; 2); (389-1); (345, 389; 15). Hence (12, 389; 100). Hence 389 is the 100^{th} number.

2.35: The infinite geometric series are convergent to the values given.

$$a + a^2 + a^3 + \ldots = \frac{a}{1-a} = 5/6 \rightarrow a = 5/11.$$

Similarly, $b + b^2 + b^3 + \ldots = \dfrac{b}{1-b} = \dfrac{6}{5} \Rightarrow b = 6/11.$

$$\therefore \quad a^2 - b^2 = (5/11)^2 - (6/11)^2 = -\frac{1}{11}.$$

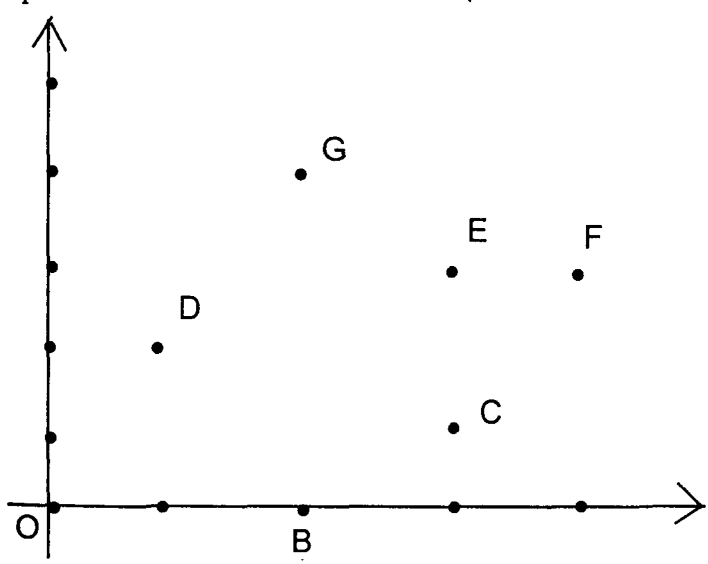
i.e., $11(a^2 - b^2) + 1 = 0$.

2.36: The centre of the circle is $(0, 0)$ and radius is 10.

When $x = 0, y = \pm 10$ and when $x = \pm 10, y = 0$.

Thus $(0, \pm 10)$ and $(\pm 10, 0)$ are four points, satisfying the conditions of the problem. Further, when $x = \pm 6, y = \pm 8$ and when $x = \pm 8, y = \pm 6$. Thus $(\pm 6, \pm 8)$ and $(\pm 8, \pm 6)$, totally eight points are also on the circle satisfying the given conditions. Apart from these 12 points, for no integral values of x, we can get integral values of y satisfying $x^2 + y^2 = 100$. Thus (a, b) has 12 solutions. i.e., The number of 'Lattice points' required is 12.

2.37: Denoting the seven given points by $0, B, C, D, E, F$ and G, from the rough sketch, we see that all the points except F are at the same distance $\sqrt{5}$ units from D.



Thus five points $0, B, C, E, G$ are concyclic. The equation to the circle is $(x - 1)^2 + (y - 2)^2 = 5$.

2.38: Let the roots be $an^2, an, a, \frac{a}{n}, \frac{a}{n^2}$

Then sum of the roots $= a(n^2 + n + 1 + \frac{1}{n} + \frac{1}{n^2}) = 40$ (A)

Sum of the reciprocals $= \frac{1}{a}(\frac{1}{n^2} + \frac{1}{n} + 1 + n + n^2) = 10$. (B)

$\therefore$ Dividing (A) by $(B), a^2 = 4$ or $a = \pm, 2$. (C)

$\therefore$ Product of the roots $= a^5 = (\pm 2)^5 = \pm 32 = -S$ i.e., $|S| = 32$.

2.39: We may write the given equation as
$y^2 + z^2 - x(3yz - x) = 0$

or $y^2 + z^2 + (3yz - x^2) - 3yz(3yz - x) = 0$

or $y^2 + z^2 + (3yz - x^2) = 3yz(3yz - x)$. This shows that if $1 \le x \le y \le z$ is an integral solution, then, $y, z, 3yz - x$ is also one such solution. Further we have,

$$\begin{aligned} 3yz - x &\ge 3yz - z \quad (\because \quad z \ge x) \\ &\ge 3y - z \quad (\because \quad 3 \le 3y) \\ &= 2z > z \end{aligned}$$

So the new solution $y, z, 3yz - x$ satisfies $y \le z < 3yz - x$ and the new solution set has a bigger element. Starting with the solution get $1, 1, 1$ by repeated application of the above process, we get $(1, 1, 2)$, $(1, 2, 5)$, $(2, 5, 29)$, $(5, 29, 433)$. as some other integral solutions. thus the given equation has infinitely many solutions in positive integers.

2.40: When n is odd. then, there will be r even numbers and $(r + 1)$ odd numbers in $1, 2, \ldots x = 2r + 1$. Of the numbers $(a_i - i)$ where a_i is odd, at least one number will be even corresponding to a_i and i both being odd.

Hence the product $(a_1 - 1)(a_2 - 2) \dots (a_x - m)$ contains at least one even number as factor proving that the product is even.

2.41: Suppose we can write
$m(2x + 3y) = 17(nx + ly) + (9x + 5y)$ for integral values of n, l and m with m, a co-prime of 17. Then clearly when $(2x + 3y)$ is divisible by $17, (9x + 5y)$ is also divisible by 17. this means we have to solve in integers the equations $2m = 17x + 9$ and $3m = 17l + 5$.

This is equivalent to getting integral solutions for
$3(17n + 9) - 2(17l + 5) = 0$ or $17(3n - 2l) = -17$ or $3n - 2l = -1$. Many solutions exist.

One solution is $n = l = -1$.

In this case, $m = \frac{(-17+9)}{2} = -4$.

Thus $(-4)(2x + 3y)^2 = -17(x + y) + 9x + 5y$. Hence $(9x + 5y)$ is divisible by 17.

When $n = 1, l = 2$, then $x = 13$; even now
$13(2x + 3y) = 17(x + 2y) + 9x + 5y$ and the result follows.

2.42: $\dfrac{14x + 5}{9}$ is an integer if $(14x + 5)$ is a multiple of 9 or 3. But as $14x + 5 = (12x + 3) + 2x + 2$, this is possible only if $2x + 2$ is a multiple of 3. (A)

Similarly $\frac{17x-5}{12}$ is an integer if $17x - 5$ is divisible by 3 and 4. This means, that $(15x - 6) + 2x + 1$ and hence $(2x + 1)$ is a multiple of 3. (B)

Thus both the given numbers $\frac{14x+5}{9}$ and $\frac{17x-5}{12}$ will be integers (for integers x) only if both $(2x+2)$ and $(2x+1)$ are divisible by 3. (C)

This is impossible as $(2x+2)$ and $(2x+1)$ are consecutive integers.

2.43: Between 4 and 9, let us introduce number of 4's followed by x number of 8's is let the number be 444...4488...89. This number then can be written as

$$\frac{444\ldots4}{2x+2}+\frac{444\ldots4}{x+1}+1$$

$$\begin{aligned}
&= \frac{4}{9}(10^{x+2}-1)+\frac{4}{9}(10^{x+1}-1)+1\\
&= \frac{1}{9}\{4\times 10^{x+2}+4\times 10^{x+1}-4-4+9\}\\
&= \frac{1}{9}\{(2\times 10^{x+1}+1)^2\} = (\frac{2\times 10^{x+1}+1}{3})^2
\end{aligned}$$

Sum of the digits in $2\times 10^{x+1}+1$ is 3. Hence it is divisible by 3.

Thus $\dfrac{2\times 10^{x+1}+1}{3}=A$ is an integer.

This proves that the number followed by insertion of equal number of 4's and 8's between 4 and 9 is a perfect square.

2.44: The result is true only if a,b,c are integers. Suppose one of a,b,c is divisible by 7, Then the given expression is divisible by 7. Suppose none of a,b,c is divisible by 7.

Then a^3,b^3,c^3 will leave remainders 11 or 6 (0 is not possible as a,b,c are not divisible by 7)

(**Note:** The possible remainders when the cube of any integer is divided by seven is 1 or 6).

As there are only two possible remainders, two of a^3, b^3, c^3 will leave the same reminder. So $a^3 - b^3$, $b^3 - c^3$ or $c^3 - a^3$ will be divisible by 7. Hence, in any case, $abc(a^3 - b^3)(b^3 - c^3)(c^3 - a^3)$ is divisible by 7.

2.45: From $\frac{1}{a} + \frac{1}{b} + \frac{1}{a+x} = 0$, we get $x = -\frac{(a+2b)}{a+b}$ and from $\frac{1}{a} + \frac{1}{c} + \frac{1}{a+y} = 0$, we get $y = -\frac{a(a+2c)}{a+c}$.
Substituting these values in $\frac{1}{a} + \frac{1}{x} + \frac{1}{y} = 0$, we get

$$\frac{1}{a} - \frac{a+b}{a(a+2b)} - \frac{a+c}{a(a+2c)} = 0$$

or $(a+2b)(a+2c) - (a+b)(a+2c) - (a+c)(a+2b) = 0$
or $-a(a+b+c) = 0$. Since $a \neq 0, a+b+c = 0$.

2.46: $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b+c} - \frac{1}{c}$

$$\therefore \quad \frac{a+b}{ab} = -\frac{a+b}{c(a+b+c)} \qquad \text{(A)}$$

$$\Rightarrow (a+b)[c^2 + c(a+b) + ab] = 0. \qquad \text{(B)}$$

$$\text{i.e., } (a+b)(c+a)(c+b) = 0. \qquad \text{(C)}$$

$$\Rightarrow a = -b \text{ or } c = -a \text{ or } b = -c. \qquad \text{(D)}$$

Consider $a = -b$ or $b = -a$.
We have $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = \frac{1}{a^3} - \frac{1}{a^3} + \frac{1}{c^3} = \frac{1}{c^3}$.

Also $\frac{1}{a^3 + b^3 + c^3} = \frac{1}{a^3 + a^3 + c^2} = 1/c^3$.

$$\therefore \quad \frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{1}{a^3+b^3+c^3}.$$

Symmetrically for $c=-a$ or $b=-c$, we get

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{1}{a^3+b^3+c^3}$$

This completes the proof.

2.47: We can prove the fact and it holds for natural numbers a_i only for $i \in \{1,2,3\ldots 51\}$

Let us consider the case that all the a_i's are natural numbers. Let S be the sum of the 50 consecutive differences.

Then $S=(a_2-a_1)+(a_3-a_2)+(a_4-a_3)+\ldots+(951-950)$

$$\Rightarrow S=a_{51}-a_1$$

(due to telescopic cancelation of terms $a_2, a_3, \ldots a_{50}$)

S is maximum when a_{51} is maximum and a_1 is minimum.

$$\therefore \quad S_{\text{max}}=\max(a_{51}-\min(a_1)).$$

But $a_{51}<142$ and is natural $\Rightarrow \max(a_{51})=141$ and $a_1>1$ and is natural $\Rightarrow \min(a_1)=2$.

$$S_{\text{max}}=139.$$

Let us assume that, among the 50 consecutive differences a_i-a_{i-1} for $i \in \{2,3,4,\ldots 51\}$, no value will occur 12 or more number of times. The S is minimum when a difference of 1 repeats 11 times exactly, a difference of 2 repeats 11 times exactly, a difference of 3 repeats 11 times exactly, a difference of 4 repeats 11 times exactly an a

difference 5 repeats 6 times exactly.

$$S_{\text{min}} = \underbrace{(1+1+\ldots 1)}_{11 \text{ times}} + \underbrace{(2+2+\ldots+2)}_{11 \text{ times}} + \underbrace{(3+3+\ldots+3)}_{11 \text{ times}}$$

$$+ \underbrace{(4+4+\ldots+4)}_{11 \text{ times}} + \underbrace{(5+5+5+\ldots+5)}_{6 \text{ times}}$$

$$S_{\text{min}} = 11 \times (1+2+3+4) + (5 \times 6) = 140$$

And $S_{\text{min}} > S_{\text{max}}$. There is some value of difference that occur at least 12 times. Hence proved.

2.48: Any perfect square ending with digit 6 only will have an odd digit in the ten's place. (A)

If a perfect square ends with $101x$, then x must be 6 only. (B)

$\therefore \quad N = \ldots 1016$ is a perfect square. (C)

But last three digits suggests N is divisible by 8.

$\therefore \quad 016$ is divisible by 8. (D)

And last 4 digits suggest that N is not divisible by 16.
$\therefore \quad 1016$ is not divisible by 16. (E)

But a perfect square, if divisible by $8 = 2 \times 2 \times 2$ should be divisible by $16 = 2 \times 2 \times 2 \times 2$. (F)

$\therefore$ As N does not satisfy this condition, N cannot be a perfect square. (G)

This completes the proof.

2.49: As $[n + \frac{99}{100}] - [n + \frac{20}{100}] \le 1$, we have

$$[n + \frac{99}{100}] - [n + \frac{20}{100}] = 0, \text{ or } 1.$$

$\therefore$ The sequence of integer functions $[n+\frac{20}{100}]$, $[n+\frac{21}{100}]$, $[n+\frac{22}{100}]\ldots[n+\frac{99}{100}]$ takes value of either a single integer n or takes values of two consecutive integers m and $m+1$. There are 80 such functions.

$\therefore$ We have either $80m = 1606$ or

$x.m+(80-x)(m+1) = 1606$ where $1 \le x \le 79, x \in 2$. As 1606 is not a multiple of 80, $80m = 1606$ has no solution in integers. Hence, there are since 'x' function integers taking values of integer x and the nest integral functions taking the values $(m+1)$. 'm' cannot less than 20 because $80 \times 20 = 1600 < 1606$ and therefore

$$x.m+(80-x)(m+1) < 1606 \text{ for } \quad \begin{array}{c} x < 20 \\ 1 \le x \le 79 \\ x \in Z. \end{array}$$

Similarly m cannot be greater than 20 because $80 \times 21 = 1680 > 1606$.

$$\therefore\ x \cdot m + (80-x)(x+1) > 1606 \text{ for } \quad \begin{array}{c} x > 20 \\ 1 \le x \le 79 \\ x \in Z. \end{array}$$

$\therefore$ m should be $20 \Rightarrow m+1 = 21$.

Hence the equation reduces to $20x + (80-x)21 = 1606$.

$\therefore$ $1680 - x = 1606 \rightarrow x = 74$.

$\Rightarrow [(n+\frac{20}{100})+\frac{73}{100}] = 20$ and $[(n+\frac{20}{100})+\frac{74}{100}] = 21$

or $[n+\frac{93}{100}] = 20$ and $[n+\frac{34}{100}] = 21$.

$\therefore$ $20+\frac{6}{100} \le n < 20+\frac{7}{100} \Rightarrow 2006 \le 100n < 2007$.

$\therefore$ $[100n] = 2006$.

2.50: If $a \neq 0(\text{mod}3)$, then
$a^4+6a^3+11a^2+6a+1 \equiv 1(\text{mod}3)$. If $a \equiv 1(\text{mod}3)$, then
$$
\begin{aligned}
a^4+6a^3+11a^2+6a+1 & \\
&\equiv a^4+11a^2+1(\text{mod}3) \\
&\equiv 1+(11\times 1)+1(\text{mod}3) \\
&\equiv 1(\text{mod}3)
\end{aligned}
$$

If $a \equiv 2(\text{mod}3)$, then $a^4+6a^3+11a^2+6a+1(\text{mod}3)$
$$
\begin{aligned}
&\equiv a^4+11a^2+1(\text{mod}3) \\
&\equiv 1+(11\times 1)+1(\text{mod}3) \\
&\equiv 1(\text{mod}3)
\end{aligned}
$$

$\therefore$ Invariably, LHS of the equation
$$a^4+6a^3+11a^2+6a+1=\frac{q(a^2-1)(c^2-1)}{a^2+c^2}$$
is always $1(\text{mod}3)$, where a and c are integers and q is the product of arbitrary non-negative powers of alternate primes. i.e., $q = 2^{b_1}\cdot 5^{b_2}\cdot 11^{b_3}\ldots$ where $b_i \geq 0$. for the equation to be true, RHS of the equation should be $1(\text{mod}3)$. Reframe the equation as
$$(a^2+c^2)(a^4+6a^3+11a^2+6a+1)=q(a^2-1)(c^2-1).$$
When both a, c are $0(\text{mod}3)$, then LHS of the above equation is $\equiv 0(\text{mod}3)$. Whereas RHS of the above equation is $\equiv 1$ or $2(\text{ mod } 3)$. When a, c are not simultaneously $0(\text{mod}3)$, then LHS $\equiv 1$ or $2(\text{ mod } 3)$; whereas RHS $\equiv 0(\text{mod}3)$; In both the cases, the above equation is invalid.

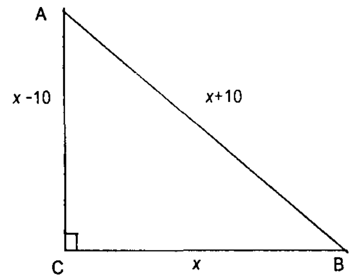
$\therefore$ There are no integral solutions in (a, c) for the given equation.

* * * * *

UNIT 2: GEOMETRY

3.01: Let one leg be x, other leg is $x-10$.

Hypotenuse is $(x+10)$.



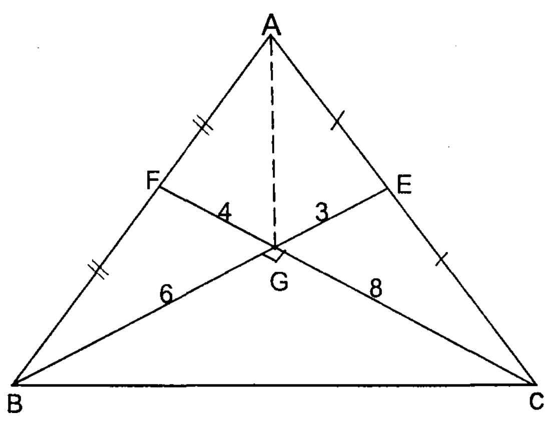
Now by Pythagoras property,

$$
\begin{aligned}
(x+10)^2 &= (x-10)^2 + x^2 \\
\therefore x^2 + 20x + 100 &= x^2 - 20x + 100 + x^2 \\
\text{i.e., } x^2 - 40x &= 0. \quad \therefore x(x-40) = 0.
\end{aligned}
$$

But $x \neq 0$, $\quad \therefore x = 40$.

$\therefore$ Hypotenuse is $x + 10 = 50\,\text{cm}$.

3.02: G is centroid.



$\dfrac{AG}{GE} = \dfrac{2}{1}$. $\quad \therefore AG = 6,\ GE = 3$.

Similarly $CG = 8, GF = 4$.
Area of $\Delta ABC = 3$ times ΔBGC.
3 times ΔBGC.

$$\begin{aligned} \text{Now,} \quad \Delta BGC &= \frac{1}{2} \times 6 \times 8 = 24 \\ \therefore \Delta ABC &= 3 \times 24 = 72\text{cm}^2. \end{aligned}$$

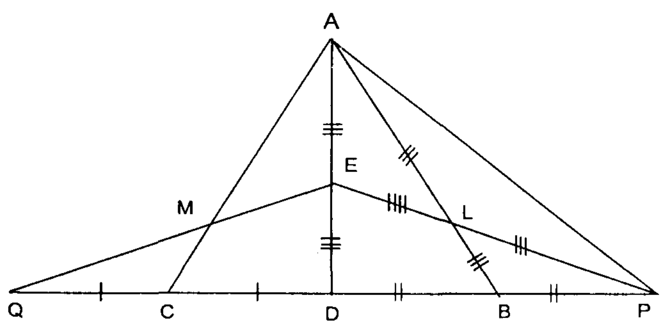
3.03: Let b, h be the base and height of the triangle respectively. The largest-part is a trapezium with sides b and $\frac{9}{10}b$ and height $\frac{h}{10}$.

$\therefore$ Area of the largest part is $\frac{1}{2}(h + \frac{9}{10}b)\frac{h}{10}$

$$\frac{1}{2}(b + \frac{9}{10}b)\frac{h}{10} = 1997.$$

$\therefore$ Area of the triangle $= \frac{1}{2}bh = \dfrac{199700}{19}\text{cm}^2$.

3.04: Join AP.



$\Delta sADP$ and ABC have the same area. PE, AB are the medians of ΔADP and hence l is the centroid of ΔADP. Area of $\Delta BPL = \frac{1}{6}$ area of ΔABC.

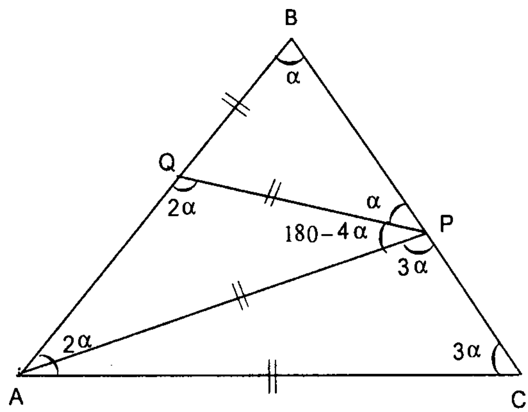
Similarly area of $\Delta CMO = \frac{1}{6}$ area of ΔABC.

Also $\triangle$s ABC and $\triangle PEQ$ have the same area.

Area of $BCEMC =$ area of $\triangle ABC - \frac{2}{6}$ area of $\triangle ABC$

$$= \frac{2}{3} \text{ area of } \triangle ABC.$$

3.05: Let $\angle B = \alpha$. Then $\angle BPQ = \alpha$ (isosceles)



$\angle AQP = 2\alpha$ (ant.angle)

$\angle QAP = 2\alpha$ (isosceles)

$\angle QPA = 180 - 4\alpha$ (A)

$\therefore \angle PAC + \angle PCA = \angle BPA = 180 - 3\alpha$ (B)

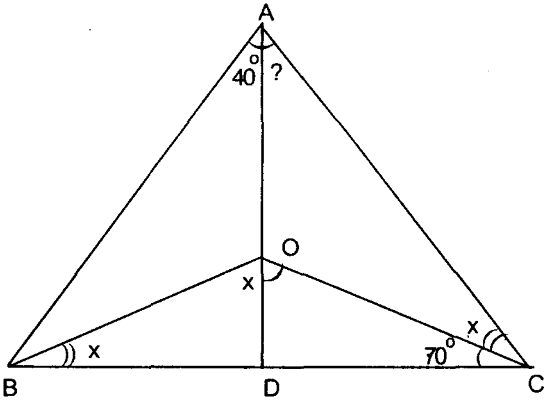
But $\angle BCA = \angle PCA = \angle BAC = \angle PAC + 2\alpha$ (isosceles $\triangle$).

$\therefore \angle PCA = 180 - 3\alpha - \angle PAC$ (C)

i.e.,$2\angle PAC = 180 - 5\alpha$

$\therefore 180 - 5\alpha + 2\alpha = 4\alpha$ or $\alpha = 25\frac{5}{7}$.

3.06: Join AO and entered it towards BC.



Then $\quad \angle COD = \angle OAC + \angle ACO (G\alpha T)$

$$\angle BOD = \angle OBA + \angle OAB$$

Adding, $\angle BOC = \angle BOC + \angle COD = \angle OAC + \angle ACO + \angle OBA + \angle OAB$

$$= \angle BAC + \angle ACO + \angle OBA$$
$$= \angle BAC + \angle ACO + \angle OCB \quad (\because \angle ABC = \angle ACB)$$

$\therefore \angle BAC + \angle ACB = 40° + 70° = 110°.$

Aliter: $\angle BOC = 90 + A/2 = 90 + 20 = 110$.

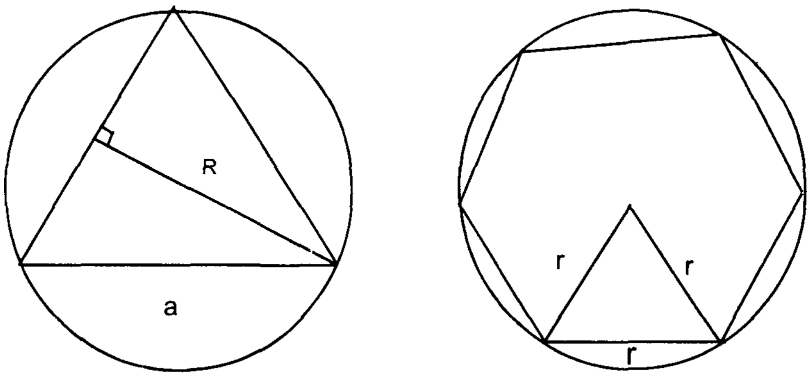
3.07: If r is the radius of the circle and 'a' is the side of the equilateral triangle inscribed therein,

$r = \frac{2}{3} \times \frac{\sqrt{3}}{2} a = \frac{a}{\sqrt{3}}$ or $a = \sqrt{3} r$. (A)

The area of the triangle $= \frac{\sqrt{3}}{4}(3r^2) = \frac{3\sqrt{3}}{4} r^2$. (B)

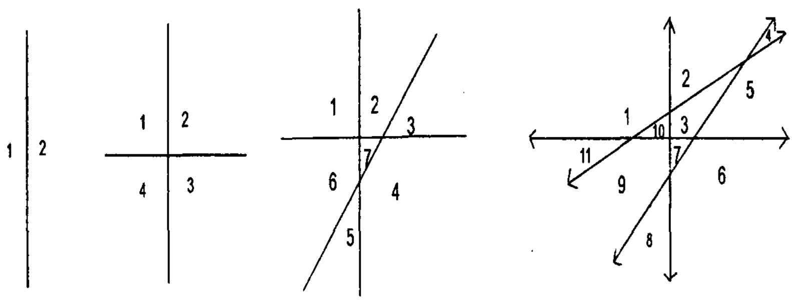
The regular hexagon inscribed in the circle is made up of six equilateral triangles with side r. Its area is

$$6 \times \frac{\sqrt{3}}{4} r^2 = \frac{3\sqrt{3}}{2} r^2. \quad \text{(C)}$$



Hence the defined ratio is $\dfrac{3\sqrt{3}}{4}r^2 : \dfrac{3\sqrt{3}}{2}r^2 = 1 : 2$.

3.08:



One st. line   Two straight lines   Three straight lines   Four straight lines

Number of st.lines	Number of regions	Difference
1	2	–
2	4	2
3	7	3
4	11	4
5	16	5
6	22	6

$\therefore$ Number of regions required under the conditions is 22.

Aliter: Number of regions under the said conditions is

$$\left[\frac{n(n+1)}{2}+1\right]=\left[\frac{6\times 7}{2}+1\right]=21+1=22.$$

3.09: Let a, b be the other two sides. Then $a+b=27$; $\therefore$

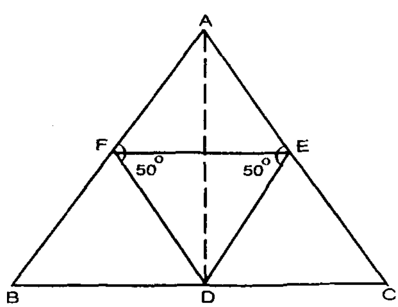
$$8=\frac{a+b+c}{2}=24;$$

$$\begin{aligned}\text{Area} &= \sqrt{s(s-a)(s-b)(s-c)}\\ &= \sqrt{24(24-a)(a-3)(24-21)}\\ &\qquad\qquad (\text{i.e., } 24-27-a)\\ &= \sqrt{72(a-3)(24-a)}\end{aligned}$$

Now $72(a-3)(24-a)$ must be a perfect square. If 'a' is the shortest side, then $a \leq 13$ (semi perimeter). Thus $a=10$ is the only possible value among the given values satisfying the conditions.

Thus, of $a=10$, area $=\sqrt{24\times 14\times 3\times 7=84}$ which is an integer. The measure of the shortest side is 10cm.

3.10: Quadrilateral $AEDF$ is cyclic and AD is the diameter of the circle circumscribing this quadrilateral which is the circumcircle of $\triangle AFE$. So $EF=2AD\sin A$. (A)



A being constant, EF is minimum when AD is minimum i.e., when D is the foot of perpendicular from A to BC.

3.11: From $\triangle BDC$,

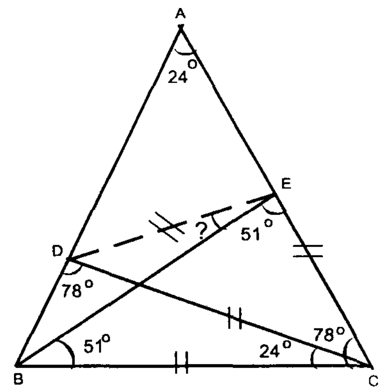
$\angle BDC = 180-(24+78°)=78°$ (A)

$\therefore\ BC = CD.$ (B)

From $\triangle BEC, \angle BEC$
$= 180 - (51 + 78)$
$= 51°$

$\therefore BC = CE$ (C)

$CD = DE$ (D)

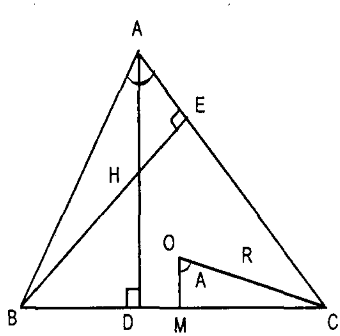


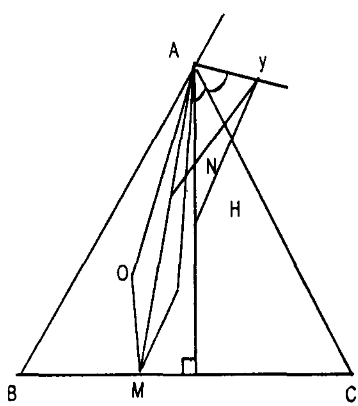
$\therefore \angle DEC = \dfrac{180 - 54}{2} = 63°.$ (E)

$\therefore \angle BED = \angle DEC - \angle BEC$

$= 63° - 51° = 12°.$

3.12: Let AH and XY intersect at N. Let O be the circumcentre of $\triangle ABC$.



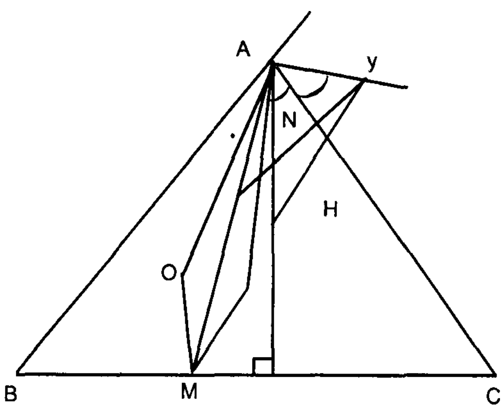


$$\begin{aligned} \angle XNH = 2\angle XAH &= 2(\angle CAH - \angle CAX) \\ &= 2(90 - \angle C - A/2) \\ &= 180 - 2\angle C - \angle A \\ &= \angle B - \angle C. \end{aligned}$$

Now $AN = \frac{1}{2}AH = OM$. Also $AN \| OM$ (both being $\perp r$.). $\therefore AOMN$ is a parallelogram. Hence, $\angle MNH = \angle OAN = \angle CAN - \angle CAO = 90 - (90 - B)$.

(Since $\angle OA = 2B) = B$. So NX and NM coincide or Y, X, M are collinear.

Note: If R is the circumradius, $OM = R\cos A$ and $\Delta ADC \sim \Delta AEH$.



$$\therefore \frac{AH}{AC} = \frac{AE}{AD}$$

$$\therefore AH = AC\,\frac{AE}{AD}$$

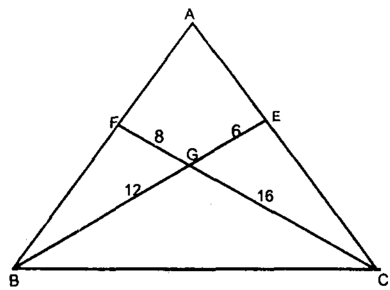
$$\text{i.e.,} \qquad AH = AC\frac{AB\cos A}{AC\sin C}$$

$$= \frac{AB}{\sin C}\cos A$$

$$= 2R\cos A$$

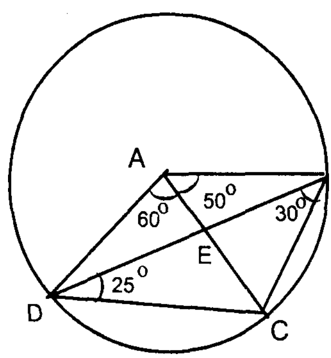
$$= 20M.$$

3.13: Since $\dfrac{BG}{GE} = \dfrac{CG}{GF} = 2$, ($G$ is the centroid), $BG = 12, CG = 16$. Area of $\triangle ABC = 3\triangle BGC$.



$$\begin{aligned} &= 3\sqrt{24 \times 4 \times 8 \times 12} \\ &= 3\sqrt{6 \times 6 \times 4 \times 4 \times 2 \times 2 \times 2 \times 2} \\ &= 288 \text{ sq. units.} \end{aligned}$$

3.14: Since the chords DC, CB, subtend twice the angle

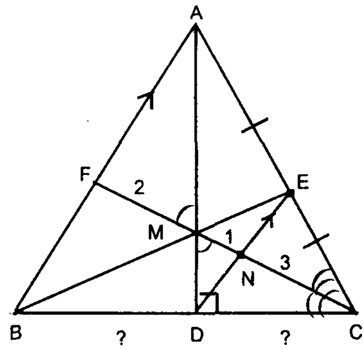


at the circumference of the circumcircle of $\triangle BCD$, viz, at B and D respectively, A is the circumcentre, i.e., A is the circumcentre of $\triangle BCD$.

So $A\hat{B}E = A\hat{D}E = \dfrac{180 - 110}{2} = 35°$.

Hence $\angle AEB = 180 - (50 + 35°) = 95°$.

3.15:



$FN = 3 = NC$. $\therefore N$ is the mid point of FC.

Also E is the said of AC. $\therefore ND \parallel AB$.

$\therefore O$ is the mid point of BC.

Thus AD is the altitude as well as the median of ΔABC.

$\therefore AB = AC$

AD is also angle bisector of $\angle A$.

$\Delta AFM \sim \Delta DNM$ since $\angle FAM = \angle MDN$

$\angle AMF = \angle DMN$. $\therefore \dfrac{AM}{MD} = \dfrac{FM}{MN} = 2/1$.

Thus M is the centroid of ΔABC.

Since CF passes through M, CF is also a median i.e., angle bisector CF is also median $\therefore CA = CB$. Thus ΔABC is equilateral.

$\therefore CF = 6 =$ altitude of equilateral angle.

Side of the equilateral triangle $= \dfrac{12}{\sqrt{13}} \left(\dfrac{\sqrt{3}}{2} a = 6 \right)$

$\therefore$ Perimeter $= \frac{12}{\sqrt{3}} \times 3 = 12\sqrt{3}$.

Method II: Consider $\triangle AFC$ and the fact that AM is the angle bisector of $\angle A$.

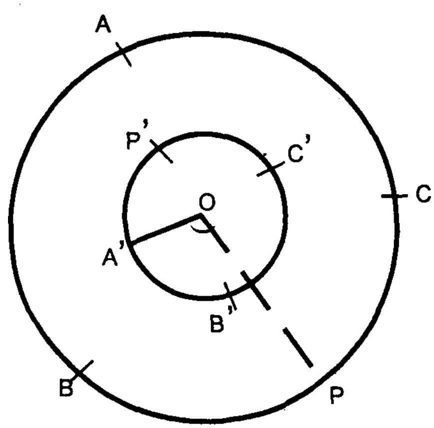
$$\frac{AF}{AC} = \frac{FM}{MC} = \frac{1}{2}, \quad AF = \frac{AC}{2} = \frac{AB}{2}.$$

Since $AB = AC$. F is the mid point of AB. Hence CF is the median, angle bisector and altitude

$$\therefore AB = CA = BC.$$

$\therefore$ The triangle is equilateral etc.

3.16:



$$\begin{aligned} \text{Then } (PA')^2 &= (OP^2) + (OA')^2 - 2OP \cdot OA' \cdot \cos\theta \\ &= R^2 + r^2 - 2rR\cos\theta \qquad (A) \\ (PB')^2 &= r^2 + R^2 - 2rR\cos(\theta - 120) \qquad (B) \\ (PC')^2 &= r^2 + R^2 - 2rR\cos(\theta + 120) \qquad (C) \end{aligned}$$

$\therefore (PA')^2 + (PB')^2 + (PC')^2$

$= 3r^2 + 3R^2 - 2rR[\cos\theta + \cos(\theta - 120) + \cos(\theta + 120)]$.

$$\begin{aligned} &= 3r^2 + 3R^2 - 2rR(\cos\theta + 2\cos\theta . \cos 120^\circ) \\ &= 3r^2 + 3R^2 - 2rR(\cos\theta - 2.\frac{1}{2}.\cos\theta) \\ &= 3R^2 + 3r^2. \end{aligned}$$

Similarly, $P'A^2 + P'B^2 + P'C^2 = 3R^2 + 3r^2$.

Method II:

O is the centroid of both the triangles ABC and $A'B'C'$. In a triangle XYZ, if G is the centroid and P is any point, then,

$PX^2 + PY^2 + PZ^2 = 3PG^2 + GX^2 + GY^2 + GZ^2$.

In ΔABC, using P' as point and 0 as the centroid

$$P'A^2 + P'B^2 + P'C^2 = 3P'O^2 + OA^2 + OB^2 + OC^2$$

$$= 3R^2 + 3r^2.$$

Similarly the other expression is $3R^2 + 3r^2$.

3.17: Area of trapezium

$$ABFE = \frac{1}{2}h\left(x + \frac{x+y}{2}\right)$$

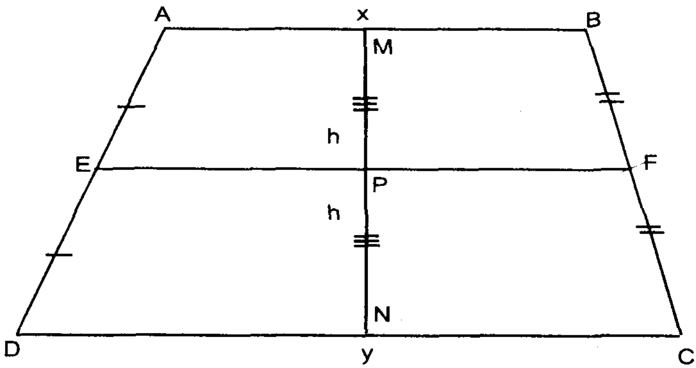
where $PM = PN = h$ and $AB = x$ and $x = y$.

For area of trapezium

$$ABFE = \frac{1}{2}PM(AB + EF) = \frac{1}{2}h(x + EF)$$

and

$$EF = \frac{(AB + DC)}{2} = \frac{x+y}{2}.$$



Similarly area of trapezium $EFDC = \frac{1}{2}h(y + \frac{x+y}{2})$.

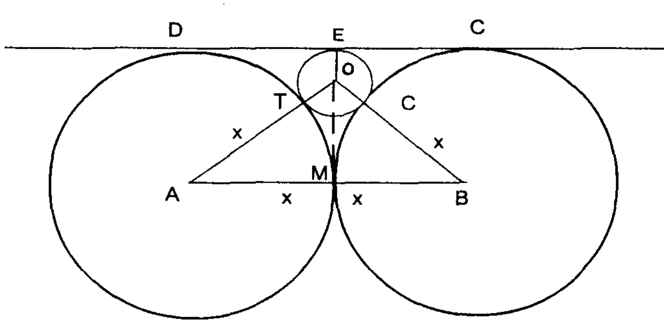
Hence, $\frac{\text{Area of } ABEF}{\text{Area of } EFDC} = \frac{\frac{3x+y}{2}}{\frac{3y+x}{2}} = \frac{5}{2}$.

$$6x + 2y = 15y + 5x.$$

$$\therefore \frac{x}{y} = \frac{13}{1} \quad \text{i.e., } x : y = 13 : 1.$$

Thus the parallel sides are in the ratio $13 : 1$.

3.18: Let the radius of the equal circles be x.



DC and ME are direct common tangents to both circles. All three circles touch DC (given).

Circles with centres A and B are equal.

So $AO = OB$ since ME bisects AB and $AM = MB$. Hence OM is the common tangent of equal circles touching at M. Also $AD = BC$.

Since DC is the tangent, AD and BC are perpendicular to DC, hence $AD \parallel BC$ and $AD = BC$. So $ADCB$ is a rectangle.

Now $MBCE$ is a square $(ME \| BC, MB \| EC, MB \| BC)$.

In $\triangle OMA$, $OM^2 + AM^2 = AO^2$.

But $MO = ME - EO = x - 4$ and $EO = 4$ as given, $AO = x + 4$ and $AM = x$.

Thus $(x-4)^2 + x^2 = (x+4)^2$

i.e., $x^2 - 8x + 16 + x^2 = x^2 + 16 + 8x$; i.e, $x = 16$.

Method -II:

Length of common tangent DC is given by,

$2\sqrt{ab} = 2\sqrt{x^2} = 2x$. (A)

Length of common tangent

$DE = 2\sqrt{ac} = 2\sqrt{4x} = 4\sqrt{x}$. (B)

Length of common tangent

$EC = 2\sqrt{bc} = 2\sqrt{x.4} = 4\sqrt{x}$. (C)

Now $4\sqrt{x} + 4\sqrt{x} = 2x(DE + EC = DC)$.

$8\sqrt{x} = 2x$

$4\sqrt{x} = x$ i.e., $\sqrt{x} = 4$

$\therefore x = 16$.

3.19: Let the angles at each vertex be $\theta_1, \theta_2, \theta_3, \cdots \theta_n$ degrees. Then the sum of these angles $= \theta_1 + \theta_2 + \cdots + \theta_n$

$$= \frac{15}{2}[n.180 - (\theta_1 + \theta_2 + \cdots + \theta_n)]$$

or $\frac{17}{2}(\theta_1 + \theta_2 + \cdots + \theta_n) = 15 \times 90n$

i.e, $17(\theta_1 + \theta_2 + \cdots + \theta_n) = 30 \times 90n$.

But $\theta_1 + \theta_2 + \cdots + \theta_n = (2n - 4)90°$.

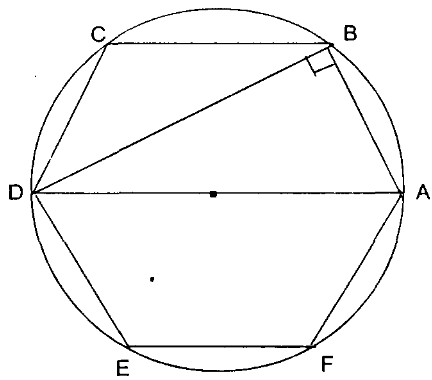
Thus $17(2n - 4)90° = 30 \times 90n$. $\therefore n = 17$.

3.20: The diagram contains 10 fundamental rectangles. We can form rectangles by combining two, three, of these. If nr is the number of rectangles obtained by combining r of them, then,

r	1	2	3	4	5	6	7	8	9	10	total
nr	10	6	2	5	4	1	0	2	0	1	31

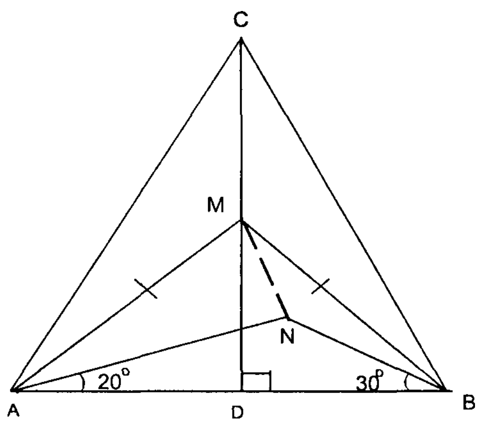
3.21: With the help of the compasses, plot the vertices A, B, C, D, E and F of a regular hexagon on the circumference of a circle of radius r.

Now AD is a diameter of the circle. AD will subtend a right angle at any point of the circle.
(except at A and D). For example $\angle ABD = 90°$. Then the two circles with A and D as their centres and AB



and DB as their radii respectively will intersect at right angles, B being a point of their intersection.

3.22: $\angle MAB = \angle MBA = 40°$ implies $MA = MB$. So



MC is $\perp r$ bisector of AB.

So $\angle ACM = \angle BCM = \dfrac{60°}{2} = 30°$

In $\Delta s AMC$ and $ANB, \angle ACM = \angle ABN = 30°$,

$\angle CAM = \angle BAN = 20°$ and $CA = BA$.

So $\triangle AMC \equiv \triangle ANB$ implying $AM = AN$. So

$$\angle AMN = \angle ANM = \frac{180° - \angle MAN}{2} = \frac{180° - 20°}{2} = 80°$$

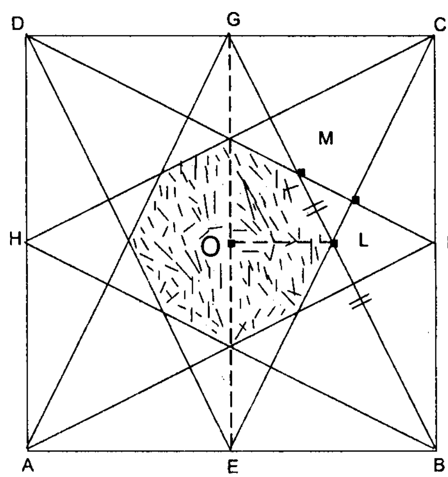
$$\angle AMD = \angle MAC + \angle ACM = 20° + 30° = 50°.$$

$$\text{So}\angle DMN = \angle AMN = \angle AMD = 80° - 50°$$
$$= 30° = \angle DCB.$$

Hence $MN \parallel BC$.

3.23: Let E, F, G, H be the midpoints of sides AB, BC, CD, DA.

Then $GFBC$ is a rectangle and 0 is the midpoint of GE.



Let GB and EC meet at L. then L is the midpoint of GB.

$\therefore OL = \frac{1}{2}EB = \frac{1}{4}$.

Let BG and DF are two median of $\triangle BCD, r$ is its centroid.

Since CO is its third median, $MO = \frac{1}{2}CO$.

But $CO = \frac{\sqrt{2}}{2}$.

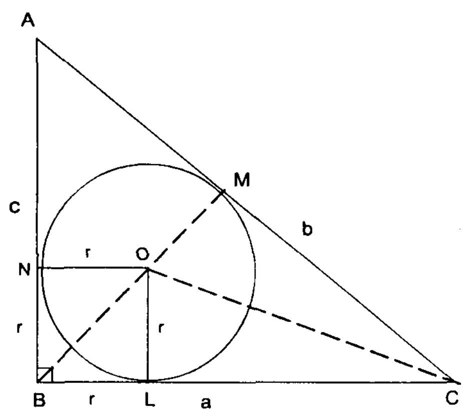
$\therefore\ OM = \frac{\sqrt{2}}{6}$.

Since $\angle MOL = \angle COF = 45°$,

the area of $\triangle OLM = (\frac{1}{2})(1/4)(\sqrt{2}/6)45 = 1/48$.

Because of symmetry, the area of the octagon is 8 times the area of $\triangle OCM = 8 \times 1/48 = 1/6$.

3.24: Let ABC be a right angled triangle with $\angle B = 90°$.



Let 0 be its incentre and L, M, N be the points of contact of the incircle with the sides a, b, c respectively.

Suppose that inradius is r. Now, as $\angle ABC = 90°$, the quadrilateral $NBCO$ is a square.

$\therefore NB = BL = r$.

Also $AM = AN = AB - NB = C - r$ and $CM = CL = BC - CL = a - r$, (equal tangents property).

$\therefore AM + CM = (C - r) + (a - r) = C + a - 2r$

$\Rightarrow r = \dfrac{(c+a) - b}{2}$

As $\angle B = 90°, b^2 = c^2 + a^2$. We have if

c and a are both odd or both even, $c^2 + a^2$ is even $\rightarrow b^2$ is even $\rightarrow b$ is even $\rightarrow (c+a) - b$ is even.

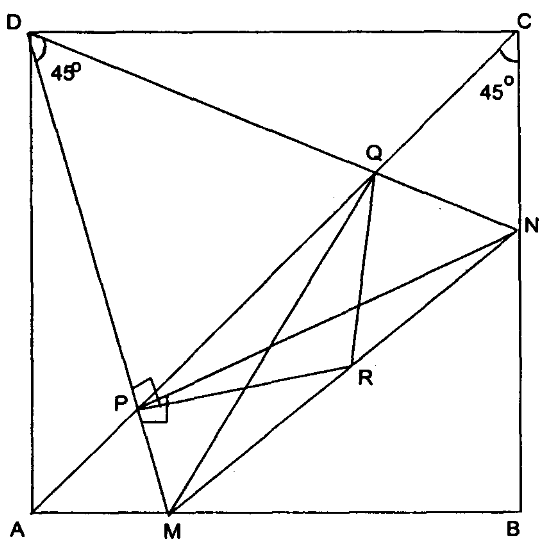
(ii) if one of c and a is even and the other is odd, $c^2 + a^2$ is odd $\rightarrow b$ is even $\rightarrow b$ is odd

$\Rightarrow (c+a) - b$ is an even number.

So, in any case, if a, b, c are integers, we have

$n = \dfrac{(c+a) - b}{2}$ = an integer.

3.25:



Join PN and QM; $\angle ACB = 45°$ and $\angle MDN = 45°$

$\therefore \angle PDN = \angle PCN$

$\therefore PNCD$ is cyclic (subtended by the same segment PN)

$\therefore \angle DCN + \angle DPN = 180°$.

But $\angle NCD = 90°$ i.e., $\angle NPD = 90°$

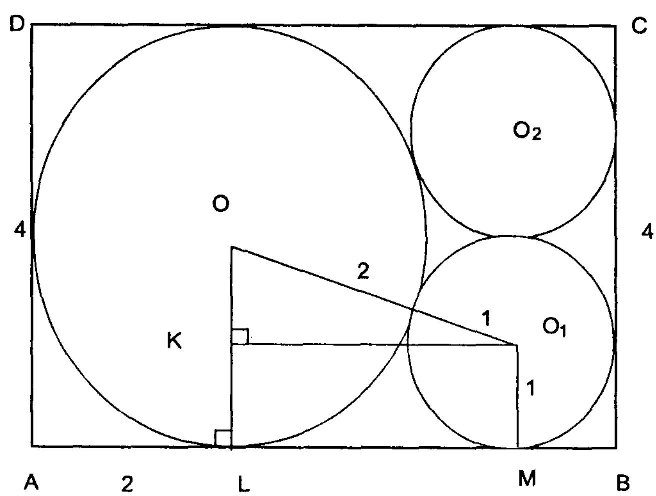
$\therefore \angle NPM = 90°$

$\therefore MN$ is the diameter of the circumside of $\triangle NPM$ with R as its centre (as R is the mid point of MN). Similarly we can show that quadrilateral $QMAD$ is cyclic. Thus $MNQM$ is a cyclic with R as its centre. $\therefore RP = RQ$..

3.26: l_1, l_2, l_3 form a triangle and there are four circles, viz, the incircle and the three excircles which touch them.

The number of circles for which all of l_1, l_2, l_3 are tangents, is 4.

3.27: Width $= AL + LM + MB$; $AD = BC = 4$ (given)



$\therefore AL = OL = 2$;

$MB = O_1M = 1;$

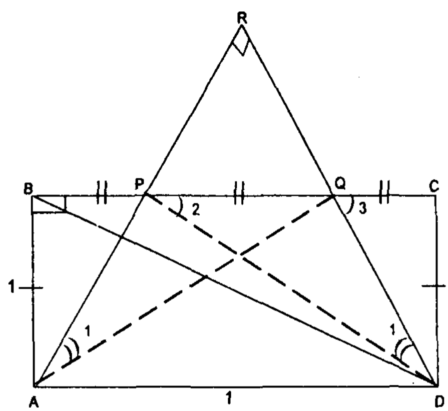
$$LM = O_1K = \sqrt{OO_1^2 - Ok^2}$$
$$= \sqrt{(2+1)^2 - (2-1)^2} = \sqrt{8}.$$
$$\therefore \text{ Width} = 2 + \sqrt{8} + 1 = 3 + 2\sqrt{2}.$$

3.28: Extend AP, DQ to meet in R.

$BP = BA$

$CQ = CD$

$\Rightarrow \angle RPQ = \angle RQP = 45°$. $\therefore \angle ARD = 90°$.



Let $AB = 1$; Then $PQ = 1/3$

$PR = RQ = 1/\sqrt{2}$ (from $\triangle PQR$)

Also $AP = \sqrt{2}$(from $\triangle PQR$)

$\therefore AR = AP + PR = 3 \cdot \frac{1}{\sqrt{2}}$

By symmetry, $\angle PAQ = \angle PDQ$

$\therefore \triangle ARQ \sim \triangle BCD$

$\therefore \angle QAR = \angle QDP = \angle DBC = \angle 1.$

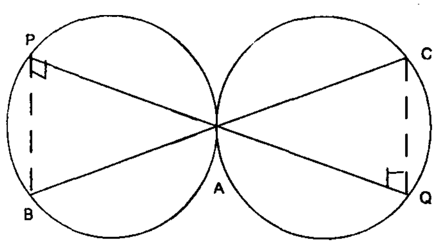
$\angle DPC = \angle PDA$ (alternate $< s$)

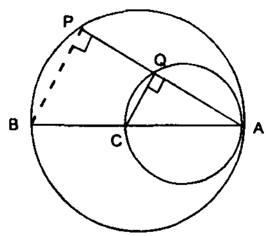
$\angle QDC + \angle PDQ + \angle PDA = 90°$

$\angle PDQ + \angle PDA = 45° = \angle DQC(\angle 1 + \angle 2 = \angle 3)$

i.e., $\angle DBC + \angle DPC = \angle DQC$.

3.29: The line through A perpendicular to the common tangent at A passes through the centres. Let it meet the circles at B, C. Then $BP \perp PA$ and $CQ \perp QA$.





This implies that $BP \parallel CQ$ (alternate $< s$).

$\therefore \Delta APB \sim \Delta AQC$

$\therefore \frac{AP}{AQ} = \frac{AB}{AC} = \frac{R}{r} = a$ constant

(where R, r are the radii of the two circles).

3.30: Given that

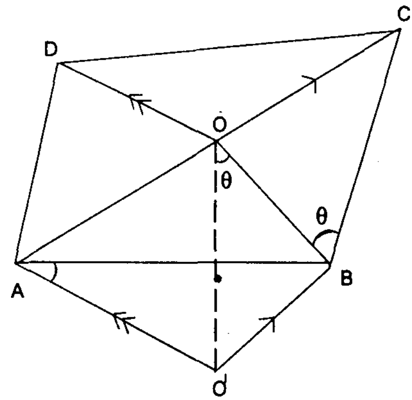
$$\angle AOB + \angle COD = \angle 180°.$$

Draw $BO' \| CO$ and $AO' \| DO$ to cut at O'.

$AB \| CD,\ AB = CD \Rightarrow \angle CDO = \angle BAO'$

and $\angle DCO = \angle ABO'$.

We get $\Delta CDO \equiv \Delta BAO'$. (I) $\therefore AOBO'$ is cyclic.



$\angle BAO' = \angle BOO'$

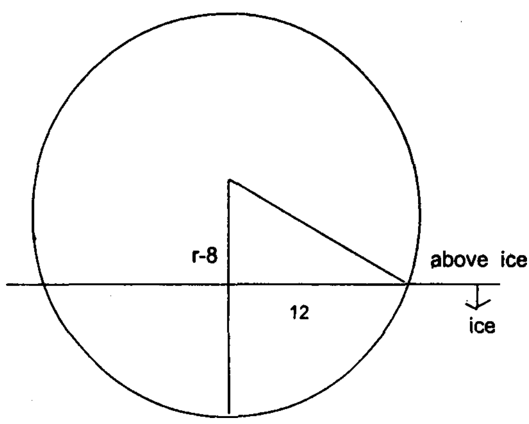
Now $CO \parallel^{l}$ and equal to BO'

$\therefore COO'B$ is a parallelogram. $\therefore OO' \| CB$.

$\therefore \angle O'OB = \angle OBC.$

From (i),(ii),(iii), $\angle CDO = \angle OBC$.

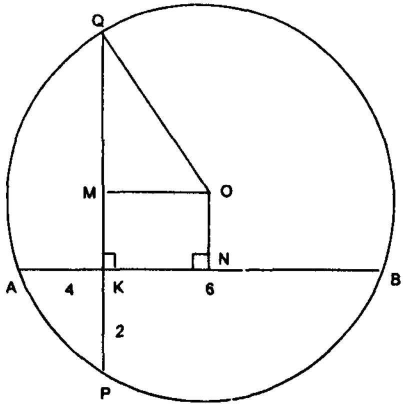
3.31: From the figure $(r-8)^2 + 12^2 = r^2$



$$\therefore r = \frac{64+144}{16} = 13.$$

i.e. the radius to the ball is 13cm.

3.32: Since $KA.KB = KP.KQ$, We get $KQ = 12$.



If we drop perpendiculars OM to PQ and ON to AB, then $KMON$ is a rectangle.

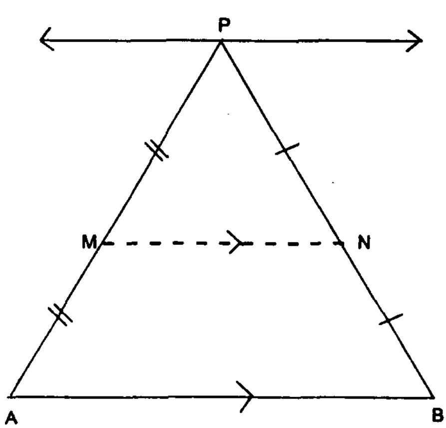
$\therefore OM = KN = KB - NB = 6 - NB = 6 - 5 = 1.$

$MK = QK - QM = 12 - 7 = 5$

$\therefore r^2 = QM^2 + MO^2 = 7^2 + 1^2 = 50$.

$\therefore$ Area $= \pi r^2 = 50\pi$. $\therefore$ The area of the circle is 50π.

3.33:



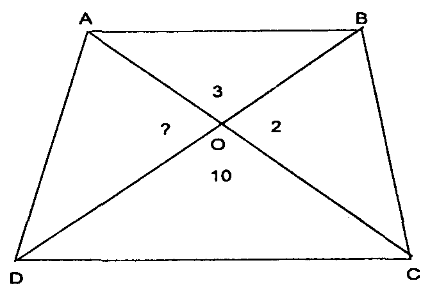
(i) Length MN will be constant because it is always $\frac{1}{2}AB$ in this case.

(ii) Since proves parallel to AB, the height of the triangle is constant and so area will be constant.

(iii) Because $MN \| AB$ and $MN = \frac{1}{2}AB$, the area of trapezoid $ABNM$ is also constant.

(iv) There is no guarantee that $PA + PB$ will be so perimeter of $\triangle PAB$ is not constant.

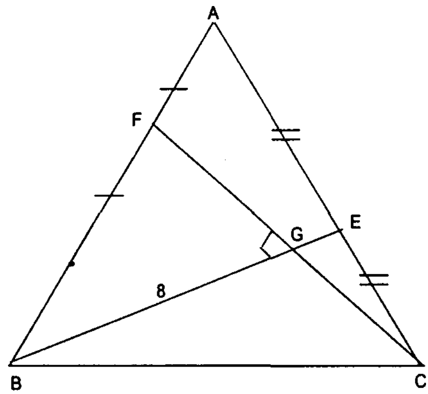
3.34: $\triangle AOD$ and $\triangle COD$ have same altitudes from D. If the area of $\triangle AOD$ is K (say) then $K : AO : OC$.



$$\text{But} \quad AO : OC = \triangle AOB : \triangle BOC$$
$$= 3 : 2$$
$$\therefore K : 10 = 3 : 2$$
$$\therefore K = 15.$$

$\therefore$ area of $\triangle AOD$ is 15.

3.35: Let G be the centroid, the meeting point of the medians



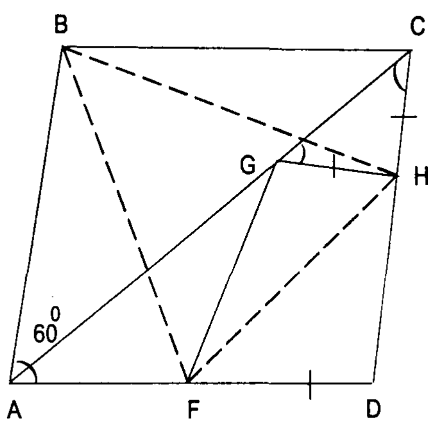
$\therefore BG = \frac{2}{3} \times BE = \frac{2}{3} \times 12 = 8\,.$

Area of $\triangle BCF = \frac{1}{2} \times CF \times BG = \frac{1}{2} \times 18 \times 8 = 72.$

Area of $\triangle ABC = 2 \times$ area $\triangle BCF$

$$= 2 \times 72 = 144.$$

3.36: We have $\angle HGC = \angle DAC = \angle ACD = 30°$



$\therefore GH = HC$ and hence $FD = HC\,.$

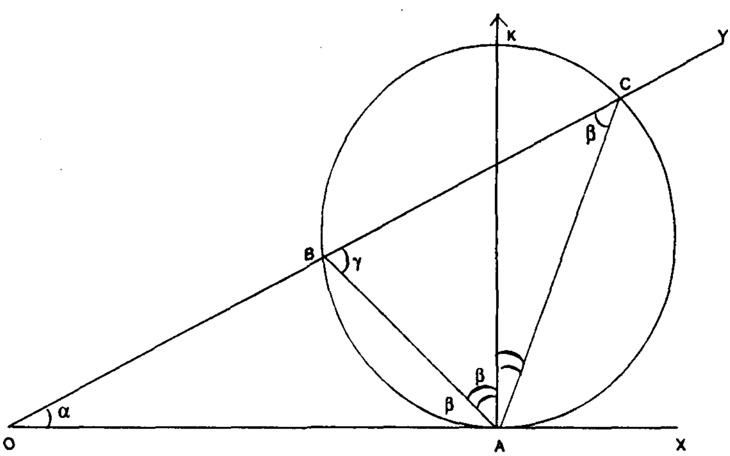
$\angle A = 60°$ implies that the rhombus $ABCD$ is divided into two equilateral triangles BDA and BDC by the diagonal BD.

In $\triangle s BDF$ and $BCH, \angle BDF = \angle BCD = 60°$ and $BD = BC, DF = CH$. Therefore the angles are congruent and $BF = BH$.

$$\begin{aligned} \therefore \angle FBH &= \angle FBD + \angle DBH \\ &= \angle HBC + \angle DBH \\ &= \angle DBC = 60°, \end{aligned}$$

hence FBH is an equilateral triangle.

3.37: Let AK be the internal bisector of the angle $\angle BAC$ made by the allowable circle. We have,



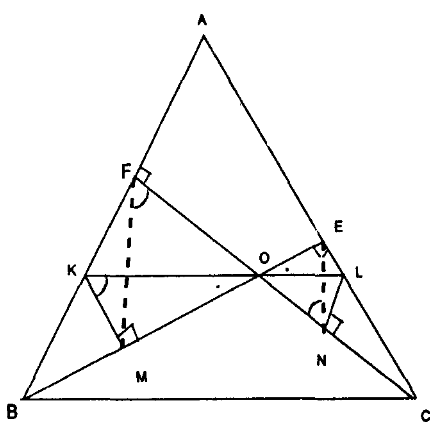
$\angle OAB = \angle ACB = \beta$ (say) (angle in the alternate segment).

$$\begin{aligned}\angle OAK &= \angle OAB + \angle BAK \\ &= \beta + \frac{1}{2}(\pi - \beta - r) \\ &= \beta + \frac{1}{2}(\pi - \beta - (\alpha + \beta)) \\ &= \pi/2 - \alpha/2 = a \text{ constant.}\end{aligned}$$

This means that the line AK is a fixed line and hence the incentres of all triangles ABC lie on this fixed line AK.

3.38: In the above figure,

$BE \perp AC$, $CF \perp AB$, $KM \perp BE$ and $LN \perp CF$ (given).



As $\angle OMK + \angle OFK = 90° + 90° = 180°$, $OFKM$ is a cyclic quadrilateral. (A)

Similarly $\angle OEL + \angle ONL = 90° + 90° = 180°$,

$\therefore OELN$ is cyclic. (B)

Again $\angle NFM = \angle OFM = \angle OKM$ (same segment). (C)

Also

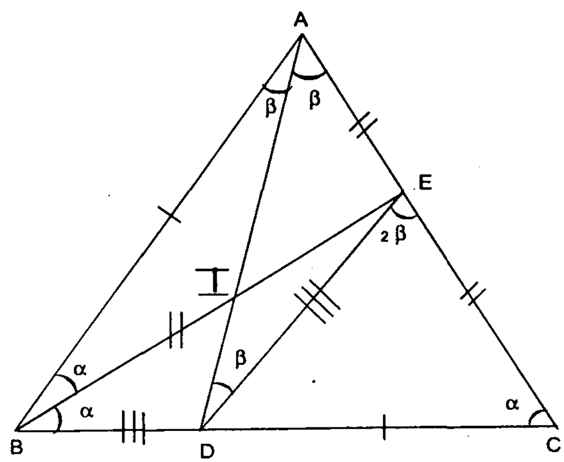
$$\begin{aligned}\angle OKM &= 90^\circ - \angle KOM \text{ (from } \triangle\text{OKM)}\\ &= 90^\circ - \angle LOE \text{ (ver. opposite angles)}\\ &= \angle \text{ (as } \triangle OLE \text{ is right } \angle d)\\ &= \angle ENO \text{ (same segment)}\\ &= \angle ENF.\end{aligned}$$

i.e., $\angle OKM = \angle ENF$. (D)

From (C) and (D), we get $\angle NFM = \angle OKM = \angle ENO$ (E)

$\therefore FM \| EN$. i.e., FM is parallel to BN.

3.39: Draw the angle bisector BE of $\triangle ABC$ to meet AC in E. Join ED. Since $\angle B = 2\angle C, \angle EBC = \angle ECB$



$\therefore EB = EC$. Now $BA = CD$, $BE = CE$

$\angle EBA = \angle ECD$ in $\triangle BEA$ and CED

$\therefore \triangle BEA \equiv \triangle CED$

$\therefore EA = ED.$

If $\angle DAC = \beta, \angle ADE = \beta$. Let D be the point of intersection of AD and BE. Now $\triangle AIB \sim \triangle DIE$.

(as $\angle BAI = \beta = \angle IDE; \angle AIB = \angle DIE$)

$\therefore \angle DEI = \angle ABI = \angle DBI$

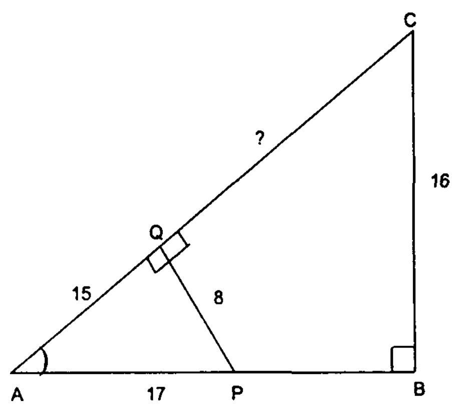
$\therefore \triangle BDE$ is isosceles and $DB = DE = EA$.

$\therefore ED \parallel AB, \quad \therefore BC = AC$
(as $BD = AE \quad \angle A = 2\angle C. \quad \therefore 5\angle C = 180$).

$\therefore C = 36°$. Also $\angle CED = \angle EAD + \angle EDA = 2\beta = A$.

$\therefore \angle A = 72°$.

3.40: $\angle Q$ and $\angle B$ one right angles.



$\therefore \triangle AQP \sim \triangle ABC$

$$\therefore \frac{AP}{AC} = \frac{QP}{BC} \qquad \text{(A)}$$

Also $AQ^2 + QP^2 = AP^2 \therefore QP^2 = 11^2 - 15^2 = 8^2$

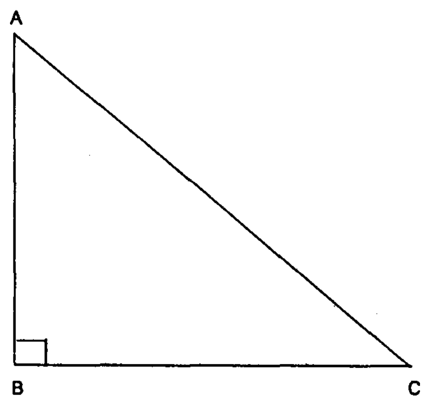
$$\therefore QP = 8. \qquad \text{(B)}$$

From (A) and (B), $\frac{17}{AC} = \frac{8}{16}$

$\therefore AC = 34$

$\therefore QC = AC - AQ = 34 - 15 = 19.$

3.41: Let ABC be a right angled triangle with $\angle ABC = 90°$



Now $AC^2 = 2AB \cdot BC$ (given).

But $AC^2 = AB^2 + BC^2 (P.T)$

$\therefore AB^2 + BC^2 = 2AB.BC$

i.e., $AB^2 + BC^2 - 2AB.BC = 0$

or $(AB - BC)^2 = 0$. $\therefore AB = BC$

(or $\angle A = \angle C = 45°$) i.e., the triangle is isosceles.

3.42: Let x be the number of sector movements. Then $75 \times x = a$ multiple of $360° = 360 \times n$.

$\therefore n = \dfrac{360 \times n}{75} = \dfrac{24n}{5} = 24$. For this to be an integer, the least value of n is 5. Hence $n = \frac{24 \times 5}{5} = 24$

i.e., The number of sector movements needed to come back to its original position is 24.

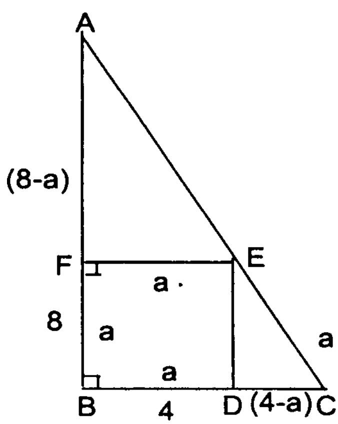
Aliter: $L.C.M$ of 75 and 360 is 1800.

$\therefore$ Number of sector movements required $= \dfrac{1800}{75} = 24$.

3.43: Total number of diagonals of an n sided polygon is $\dfrac{n(n-3)}{2} = 44$ (given) i.e., $n^2 - 3n - 88 = 0$.

$\therefore (n-11)(n+8) = 0$. $\therefore n = 11$. i.e., Number of sides of the polygon is 11.

3.44:



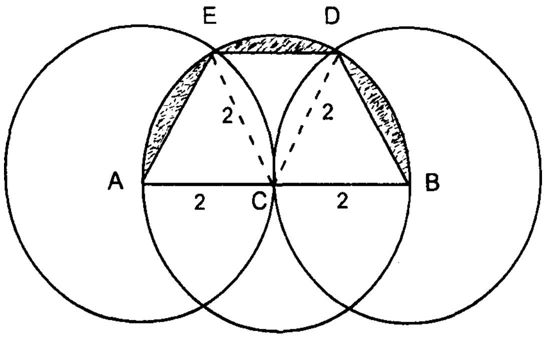
Let the square inscribed be one with side a. Then area ΔABC = area $\Delta AFE+$ area $\Delta EDC+$ area of square $BDEF$.

$$\therefore \frac{1}{2} \times 4 \times 8 = \frac{a(8-a)}{2} + \frac{a(4-a)}{2} + a^2$$

i.e., $32 = 12a$ or $a = 8/3$.

Hence the side of the inscribed square is $8/3$ units.

3.45:



ΔAEC and ΔCDB are both equilateral and congruent with sides equal to 2 units. (A)

ΔDCE is also an equilateral triangle with side 2 units (B)

$\therefore ED = 2$; But $AB = AC + CB = 4$

$\therefore$ Area of $ABDE = \Delta ACE + \Delta CED + \Delta BCD$.

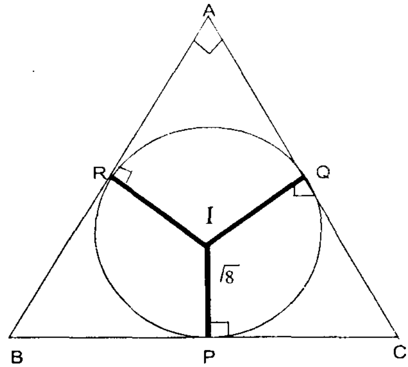
$$= 3 \times \tfrac{1}{2} \times \left(\frac{\sqrt{3}}{2} \times 2\right) \times 2 = 3\sqrt{3}\ sq.units.$$

Note: $ABDE$ is a trapezium.

$$\therefore \text{ Area } = \frac{1}{2}h(a+b) = \frac{1}{2} \times (\frac{\sqrt{3}}{2} \times 2)(6) = 3\sqrt{3}.$$

3.46: Let the incircle touch the sides BC, CA, AB in P, Q, R respectively.

Now $\angle BAC = 90° \therefore AQIR$ is a cyclic quadrilateral and $AQIR$ is a square of side $IR =$ in radius $= r$ (sq).

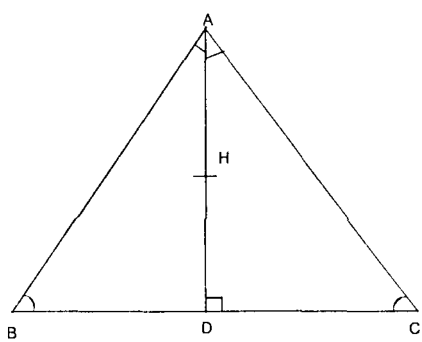


As $IP = r = \sqrt{8}, AI$ being the diagonal of the square $AQIR, AI = IR \times \sqrt{2} = \sqrt{8} \times \sqrt{2} = 4$ units.

3.47: $\triangle ABC$ is isosceles with $\angle B = \angle C$ (A)

Let AD be the altitude from A. the orthocentre H lies on AD. (B)

Also AD bisects the vertical angle $\angle A$. It contains the incentre I. (C)



Further, AD bisects perpendicularly BC. It contains the circumcentre S and (D)

Being a median, it also contains the centroid G. (E)

Hence H, I, S and G are collinear.

Note: These four points will coincide when $\triangle ABC$ is equilateral.

3.48: The parabola can cut the circle in a maximum of 4 points. When the parabola does not touch the boundary of the paper as in the above fig (a), we get the total number of regions as 6. When the boundary of the paper touches

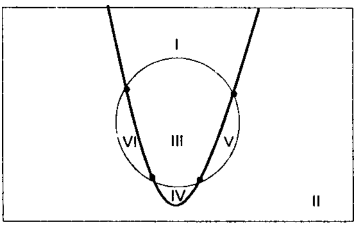


fig (a)

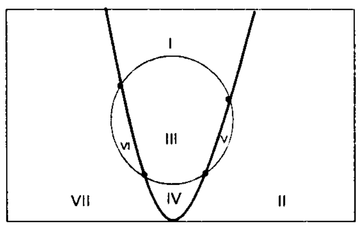
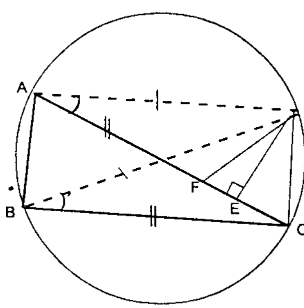


fig (b)

the parabola as in (b) the maximum number of regions is 7.

3.49: Let F be the point on AC such that $AF = BC$.



Now $AD = BD$ (D is the mid point of ACB)

$AF = BC$ (By cons)

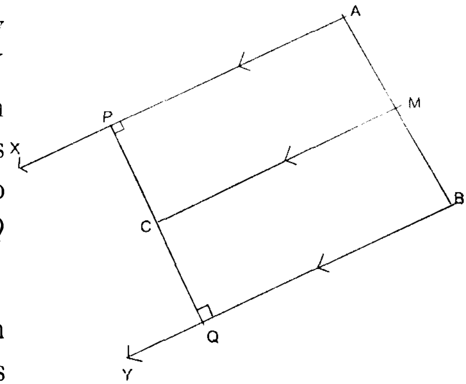
$\angle DAF = \angle DBC$ (same segments)

$\therefore \triangle AFD \equiv \triangle BCD$. $\therefore FD = CD$.

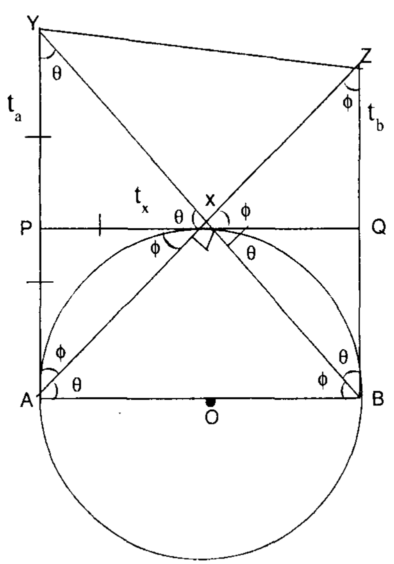
Now $AE = AF + FE = BC + EC = EC + CB$.

3.50: Find the midpoint M of AB. Join C to M and draw lines AX and BY parallel to MC. From C, drop perpendiculars to AX and BY to meet then at P and Q respectively.

Draw the circle Σ with centre C and radius CP. Then Σ is the required circle.



Proof: Since $AM = MB$ and $AP \parallel MC \parallel BQ$, We have $CP = CQ$ (equal intersects property). $\therefore$ AX and BY are tangents (parallel) to the circle.



3.51: AB is a diameter.

$\therefore\ \angle AXB = 90^\circ$

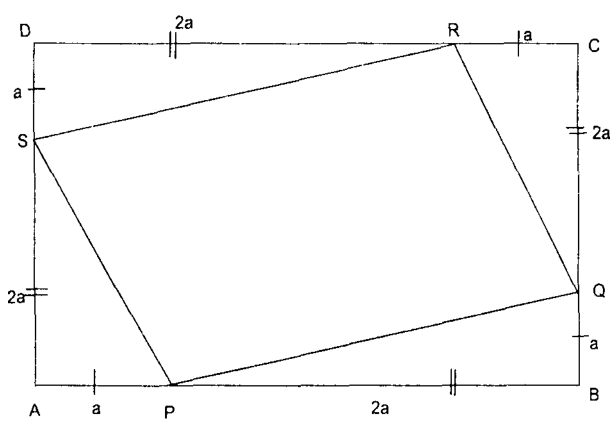
$$\begin{aligned}
\text{Let } \angle XAB &= \theta \text{ and } \angle XBA = \phi \\
\text{Now } \angle XAP &= \phi, \text{ (alt reg)} \\
\angle AXP &= \phi = \angle ZXQ \text{ (vert -- opposite)} \\
\angle BXQ &= = \theta = \angle YXP \\
\angle AYB &= \theta, \angle AZB = \Phi, \angle XBQ = \theta
\end{aligned}$$

In the right angled triangle AXY (as $\theta + \phi = 90$), we have $PQ = PX = PY$. $\therefore P$ is the midpoint of AY.

Similarly from $\triangle BXZ, QX = QB = QZ$. $\therefore Q$ is the mid point of BZ.

Hence YZ, PQ, QB concur at a point. If X is the mid point of the arc AB, then YZ, PQ, AB are parallel.

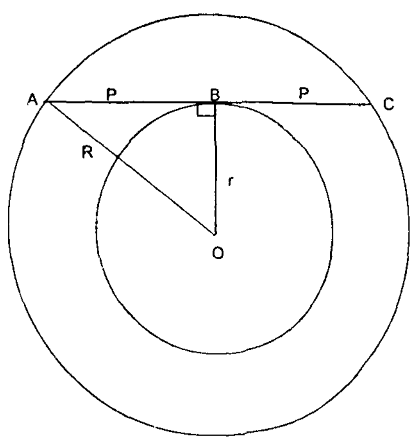
3.52: Let $AP = QB = CR = DS = a$ and $PB = QC = DR = AS = 2a$ units respectively.



$$\text{Area of square } ABCD = (3a)^2 = 9a^2 \text{ of units}$$

$$\begin{aligned}\text{Area of parallelogram } PQRS &= \text{area of square} - \text{area of} \\ &\quad\ 4 \text{ congruent}\triangle\text{ s} \\ &= 9a^2 - 4 \times (\frac{1}{2} \times a \times 2a) \\ &= 5a^2 \text{ of units} \\ &= \frac{5}{9} \times (9a^2) \text{ sq.units} \\ &= \frac{5}{9} \times \text{ area of square } ABCD\end{aligned}$$

3.53: Let the radii of the two concentric circles be R and r units respectively where $R > r$.



The line perpendicular to the chord drawn from the centre of the circle bisects the chord.

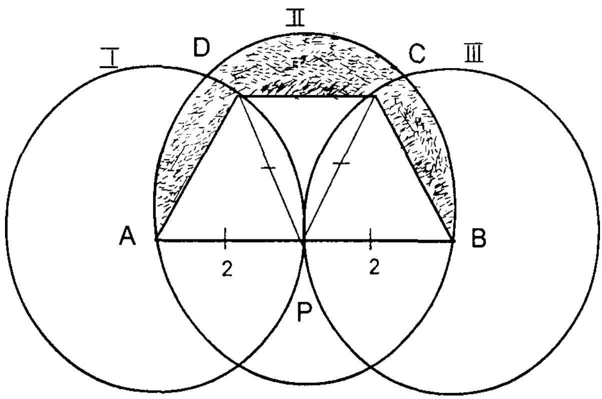
Now $R^2 - r^2 = P^2 (\triangle OAB)$.

Multiplying by π, $\pi R^2 - \pi r^2 = \pi P^2$.

i.e., Area of larger circle $-$ area of smaller circle $= \pi p^2$.

$\Rightarrow$ Area of the shaded position $= \pi p^2$.

3.54:



P is the mid point of $AB \Rightarrow PA = PB$.

But from circle $I, PA = AD$ from circle $II, PA = PD$.

$$\therefore PA = AD = PD.$$

Similarly from circles II and III .

$$PB = PC = BC \Rightarrow CD = PD = PC.$$

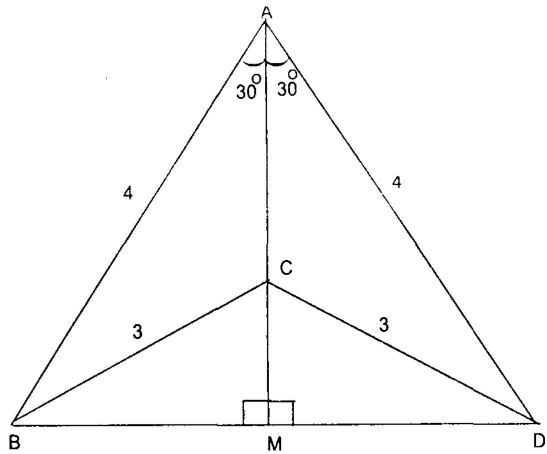
$\therefore$ Shaded area $=$ area of semicircle -area of trapezium.

$$= \frac{\pi(2)^2}{2} - \frac{1}{2} \times (\frac{\sqrt{3}}{2})[2+4]$$

$$= 2\pi - 3\sqrt{3}.$$

3.55: $\triangle ACB \equiv \triangle ACD$ (by symmetry)

$\therefore \angle BAC = \angle DAC = 30°$ Join BD and extend AC to meet BD at M .



$\angle ABD = \angle ADB$ (isosceles $\triangle$).

$\triangle ABM \equiv \triangle ADM$ (As A).

$\angle CMB = \angle CMD = 90°$.

Consider $\angle AMB$:

The angles are $30°, 60°, 90°$. $\therefore$ The sides opposite to these are in the ratio $1 : \sqrt{3} : 2$.

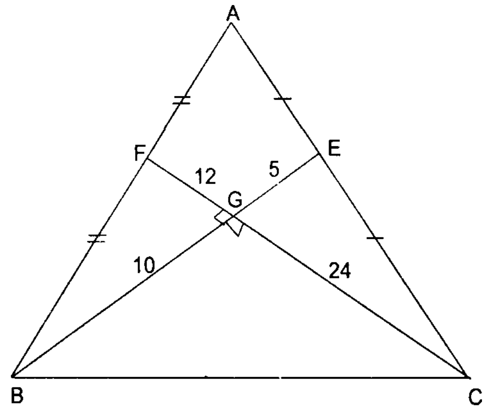
$$BM = 2 \text{ and } AM = 2\sqrt{3}.$$

From right $\triangle BCM, CM^2 = BC^2 - BM^2 = 3^2 - 2^2$.

$$\therefore CM^2 = 5 \Rightarrow CM = \sqrt{5}.$$

$$\therefore AC = AM - CM = 2\sqrt{3} - \sqrt{5}.$$

3.56: Medians intersect within the triangle and the centroid divides the median in the ratio $2 : 1$.



$$\therefore BD = 10;$$

Area of $\triangle BFC = \frac{1}{2} \times FC \times BG$.

$$= \frac{1}{2} \times 36 \times 10 = 180\text{cm}^2.$$

$\therefore$ Area of $\triangle ABC = 2\times$ area of $\triangle BFC$.

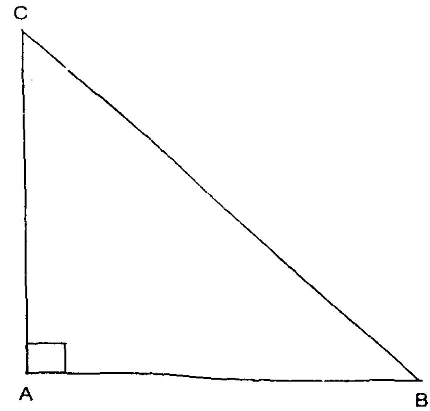
$$= 2 \times 180 = 360°\text{cm}^2$$

(as a median divides a triangle into two equal parts).

3.57: For points P on AC, we have $AP + BP + CP = AC + BP$ is minimized when P is the of the perpendicular from B to AC i.e., P coincides with A. (A)

The minimum value is $AC + AB$ (B)

For points on AB, again for a similar reason, $AP + BP + CP = AB + CP$ is minimized when P is the foot of the perpendicular from C to AB i.e., P coincides with A (C)



The minimum value is $AB + CA$. (D)

For points P on $BC, AP + BP + CP = PA + BC$. This is minimized when P coincides with the foot of the perpendicular N from A to BC. The minimum value is $AN + BC$. (E)

Now we need to check if $AN + BC > AB + AC$. (F)

$$(AN + BC)^2 - (AB + AC)^2 = AN^2 + BC^2 - AB^2 - AC^2$$

$$= AN^2 > 0 \qquad \text{(G)}$$

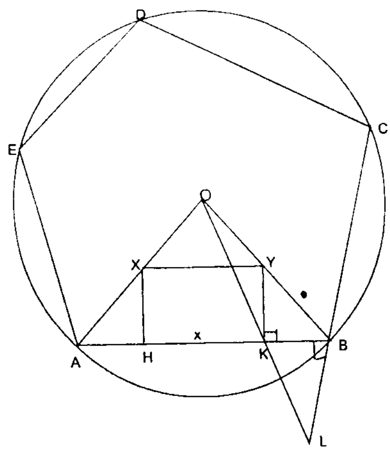
$\therefore AN + BC > AB + AC$. (H)

Hence $PA + PB + PC$ is minimized when P coincides with A.

3.58: Let O be the centres of the circle passing through the vertices of the regular pentagon $ABCDE$.

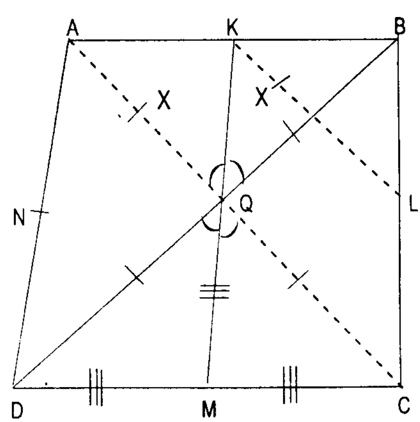
Draw BL perpendicular to BA such that

$BL = \frac{1}{4}BA$. Join OL to cut AB at K.



Draw KY perpendicular to AB cutting OB in Y. Draw YX parallel AB meeting OA at X. Then YX contains the centres of the three circles touching the side AB. the other circles are got of similar constructions. The construction works because $YXHK$ is a rectangle inscribed in $\triangle OAB$ with sides in the ratio $4:1$.

3.59:



$$\begin{aligned}
\triangle QMD &\equiv \angle QMC(SSS) \\
\therefore \angle MQD &= \angle MQC \\
\text{Also}\triangle QAK &\equiv \triangle QBK(SSS) \\
\therefore \angle KQA &= \angle KQB \\
\text{Also}\angle KQB &= \angle MQD(\text{ver.opposite}) \\
\therefore \angle AQK &= \angle CQM
\end{aligned}$$

$\therefore AQC$ is a straight line.

The diagonals AC and BD meet at Q and as $QA = QB = QC = QP$, we have the diagonals bisect each other. $\therefore ABCD$ is a parallelogram. Again as $AC = BD, ABCD$ is a rectangle.

Also $KL = MN = \frac{1}{2}AC = \frac{1}{2}BD = LM = NK$.

But by hyp(C), $\dfrac{LK}{LM} = \dfrac{CD}{CB}$.

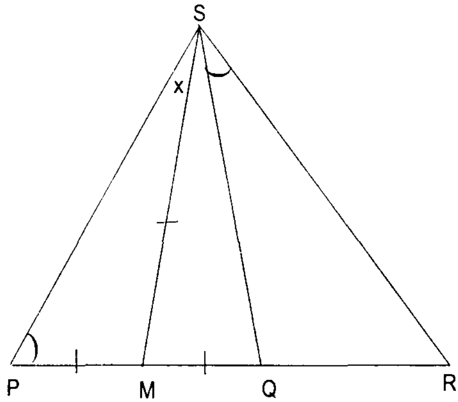
This means that $CB = CD$

$\therefore ABCD$ is a square.

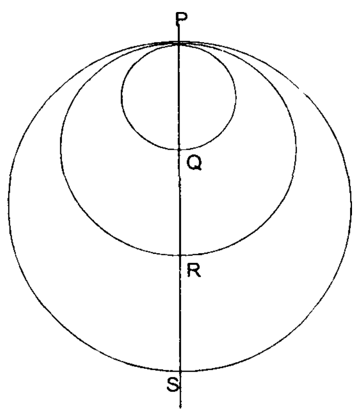
3.60: As $PM = MQ = MS$, the point M is the circumcentre of $\triangle PSQ$.

$RQ.RP = RS^2$ i.e., the tangent at R to the circle PSQ touches the circle at S.

$$\begin{aligned}
\therefore \angle QSR &= \text{angle in the alternate segment} \\
&= \angle RPS \\
&= \angle MPS = \angle MSP \quad (\text{as} MS = MP)
\end{aligned}$$



3.61: Let PQ, PR and PS be the diameters of the three circles and call then



D_1, D_2 and D_3 respectively.

Then $\dfrac{\pi D_2^2}{4} = \dfrac{1}{2}\left[\dfrac{\pi}{4}(D_1^2 + D_3^2)\right]$

$\Rightarrow 2D_2^2 = D_1^2 + D_3^2$

Given $D_1 = PQ = 1, D_3 - D_2 = RS = 0.5$

$\therefore 2(D_3 - 0.5)^2 = 1 + D_3^2$ or $D_3^2 - 2D_3 + 0.5 = 0$

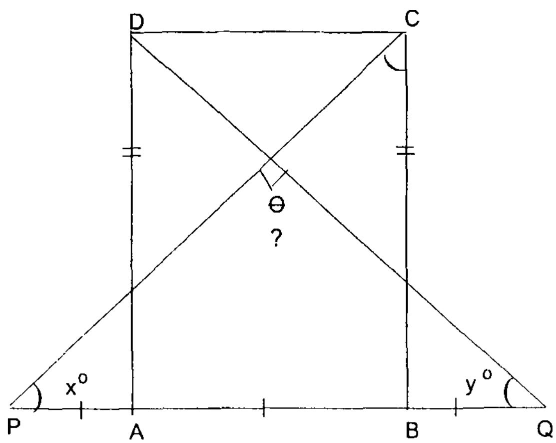
$\therefore D_3 = \dfrac{2 \pm \sqrt{4+2}}{2} = 1 \pm \frac{\sqrt{6}}{2}$

$\therefore$ Length $QR = D_2 - D_1 = (D_3 - 0.5) - 1$

$= D_3 - 1.5 = -\dfrac{1}{2} \pm \dfrac{\sqrt{6}}{2}$

As QR is not negative, $QR = \dfrac{\sqrt{6}-1}{2}$

3.62: Let CP and DQ meet at 0.



Now $AD = BC = 2AB$ (given)

$\therefore$ Both $DsPBC$ and DAQ are isosceles

$(BP = BC, AQ = AD)$.

Let $\angle BPC = x^\circ$ and $\angle AQD = x^\circ$.

From $\triangle POQ, \angle POQ = 180 - x^\circ - y^\circ$.

But $\angle PBC = 180 - 2x$ and $\angle DAQ = 180 - 2y^\circ$

But $\angle PBC + \angle DAQ = 180^\circ$ (From $\|$ D)

$\therefore 180 - 2x^\circ + 180 - 2y^\circ = 180^\circ$

i.e., $x + y = 90$

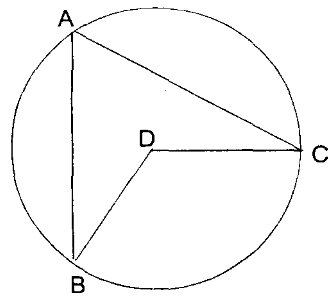
Hence $\angle POQ = 180° - 90° = 90°$;

i.e., PC and QD are perpendicular to one another.

3.63: If C is the centre of the circular path, then, $AB + BC + CD + DA < 2r + r + r + 2r$.

i.e., $< 6r$ (A)

where r is the radius of the circle. The circumference $= 2\pi r > 6r$.



Hence the distance travelled by the child will be longer than that travelled by the grand father.

When C is not the centre, there are possibilities for $AB + BC + CD + DA$ to be greater than $2\pi r$.

3.64: Join AN and produce it to meet BC at X.

In ΔAMN and DOM,

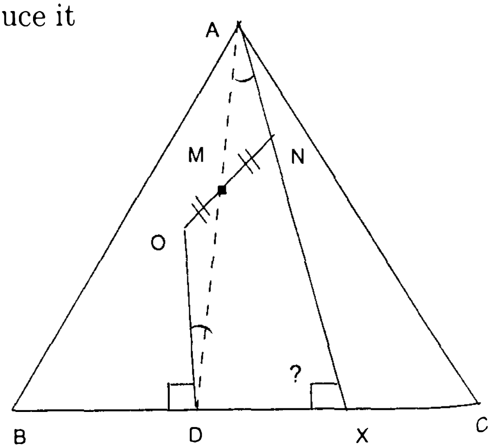
$OM = MN$ (given)

$AM = MD$ (given)

$\angle AMN = \angle DMO$

(Ver.oppo $<s$)

$\therefore \angle ODM = \angle MAN$



But these are alternate angles; $\therefore OD \parallel AN$.

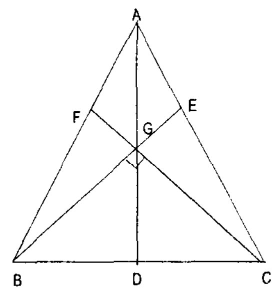
i.e., $OD \parallel AX$.

But $OD \perp BC$ (O is the circumcentre)

$$\therefore AX \perp BC$$

i.e., N lies on AX which is the altitude from A.

3.65: Let AD, BE, CF be the medians through the vertices A, B, C of $\triangle ABC$.



Using cosine formula.

$$AB^2 = AD^2+BD^2-2AB.BD\cos A\hat{D}B$$

$$AC^2 = AD^2+DC^2-2AD.DC.\cos A\hat{D}C.$$

But $\cos A\hat{D}C = -\cos A\hat{D}B$ and

$$AC^2 = AD^2 + DC^2 - 2AD.DC\cos A\hat{D}C.$$

But $\cos A\hat{D}C = -\cos A\hat{D}B$ and $BD = DC = \frac{1}{2}BC$.

Hence $AB^2 + AC^2 = 2AD^2 + 2(\frac{1}{2}BC)^2$ (A)

$\therefore 4AD^2 = 2AB^2 + 2AC^2 - BC^2$ (B)

Similarly $4BE^2 = 2BC^2 + 2AB^2 - AC^2$ (C)

and $4CF^2 = 2CA^2 + 2BC^2 - AB^2$ (D)

As G divides each median in the ratio $2:1$,

$$BG^2 = \frac{4}{9}BE^2 = \frac{1}{9}(2AB^2 + 2BC^2 - CA^2) \qquad \text{(E)}$$

$$CG^2 = \frac{4}{9}CE^2 = \frac{1}{9}(2AC^2 + 2CA^2 - AB^2) \qquad \text{(F)}$$

Since $\angle BGC = 90°$ (given)

$$\frac{1}{9}(2AB^2 + 2BC^2 - CA^2) + \frac{1}{9}(2BC^2 + 2CA^2 - AB^2)$$
$$= BC^2 \qquad \text{(G)}$$

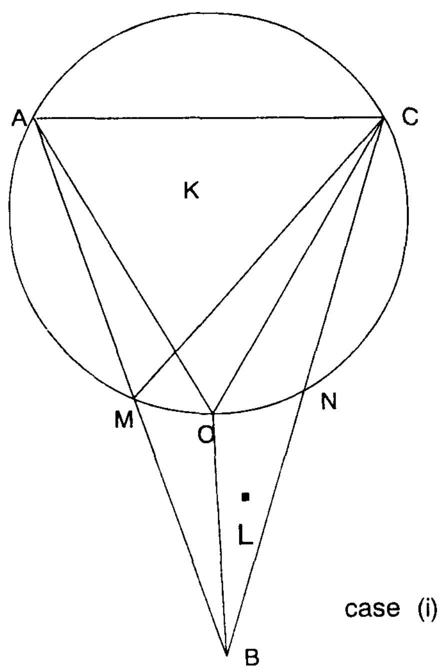
i.e.,
$$2AB^2 + 2BC^2 - CA^2 + 2BC^2 + 2CA^2 - AB^2$$
$$= 9BC^2 \qquad \text{(H)}$$

i.e., $5BC^2 = AB^2 + AC^2$ (I)

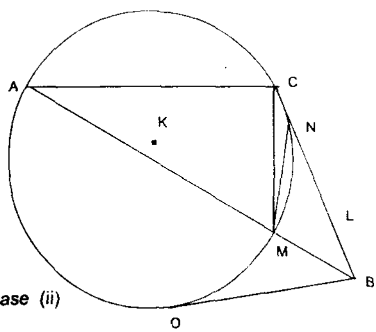
$\therefore$ Now $5BC^2 = 3 + 2 \therefore BC = 1\,\text{cm}$.

$\therefore$ Third side BC is 1cm.

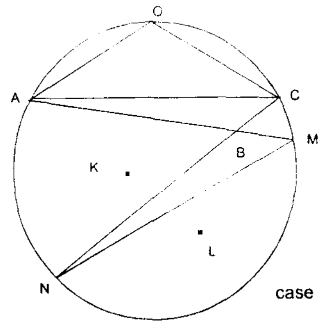
3.66:



There are three possible configurations.



case (ii)



case (iii)

Case(i):

$$\begin{aligned}\angle MCB &= \angle MCO + \angle OCA\\ &= \angle MAO + \angle OBC\\ &= \angle MBO + \angle OBC\\ &= \angle MBC\end{aligned}$$

Case(ii):

$$\begin{aligned}\angle MCB &= \angle OCB - \angle OCB\\ &= \angle OBC - \angle OAM\\ &= \angle OBC - \angle OBA\\ &= \angle MBC\end{aligned}$$

Case(iii):

$$\begin{aligned}\angle MCB &= \angle MCO - \angle OCB\\ &= (180^\circ - OAM) - \angle OBC\\ &= 180^\circ - \angle OBA - \angle OBC\\ &= \angle MBC\end{aligned}$$

Thus in all the three case.

$$\angle MBC = \angle MCB$$

We now show that L is the circumcentre of $\triangle MNB$.

Case(i):

$$\begin{aligned}\angle MLN &= \angle MKN \\ &= 2\angle MCN \\ &= 2\angle MBN\end{aligned}$$

Also $LM = LN$ $\therefore$ L is the circumcentre of $\triangle MNB$.

Case(ii):

$$\begin{aligned}\angle MLN &= \angle MKN \\ &= 2\angle MAN \\ &= 2\angle MCB \\ &= 2\angle MBN\end{aligned}$$

As $LM = LN$. So L is the circumcentre of $\triangle MNB$.

Case(iii):

$$\begin{aligned}\angle MLN &= \angle MKN \\ &= 2\angle MCN \\ &= 2\angle MCB \\ &= 2\angle MBC \\ &= 360^\circ - \angle MBN\end{aligned}$$

Also $LR = LN$. $\therefore$ So L is the circumcentre of $\triangle MNB$.

So, finally,

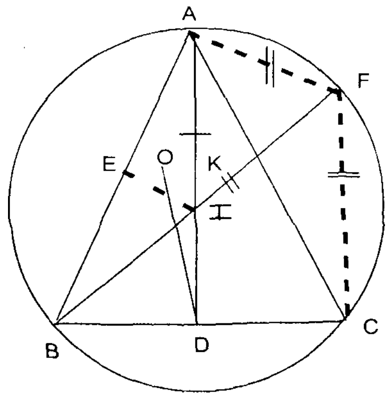
$$\angle MLB = \angle MNB$$

$$= 2\angle BAC$$

This gives $\angle MBL = 90^\circ - \angle BAC$.

This means that BL and AC are perpendiculars.

3.67:



Extend BI to meet the circumcircle in F.

Then $FI = FA = FC$. (A)

Applying ptolemy's theorem to the cyclic quadrilateral $ABCF$,

$AB.CF + AF.BC = BF.CA$ (B)

$\therefore CF(c+a) = BF.b$ i.e., As $b = \dfrac{a+c}{2}$,

$CF(a+c) = BF\frac{(a+c)}{2} \Rightarrow BF = 2CF = 2IF$ (C)

Hence I is the mid point of BF.

Also $OF = OB$ (OD is the circumcentre).

$\therefore OI$ is perpendicular to BF. (D)

Now $AK = \dfrac{bc}{a+c} = \dfrac{bc}{2b} = c/2 = AE$ (E)

Since AI bisects $\angle A, \triangle AIE \cong \triangle AIK$.

$\therefore IE = IK$ (F)

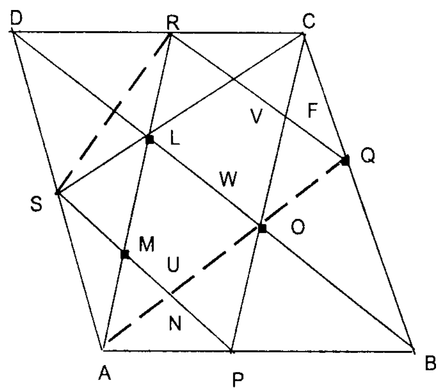
Similarly $\triangle CID \cong \triangle CIK$.

$\therefore ID = IK$. (G)

$\therefore ID = IK = IE$. (H)

i.e., I is the circumcentre of $\triangle DKE$.

3.68: We have $QR = BD/2 = PS$.



Since AQR and CSP are both equilateral and $QR = PS$ they are congruent triangles.

$$\therefore AQ = QR = RA = CS = SP = PC.$$

Also $\angle CEF = 60° = \angle RQA$.

Hence $CS \parallel QA$.

Now $CS = AQ$ implies that $CSQA$ is a parallelogram.

In particular, $SA \parallel CQ$ and $SA = CQ$

$AD \parallel BC$ and $AD = BC$

i.e., $ABCD$ is a parallelogram.

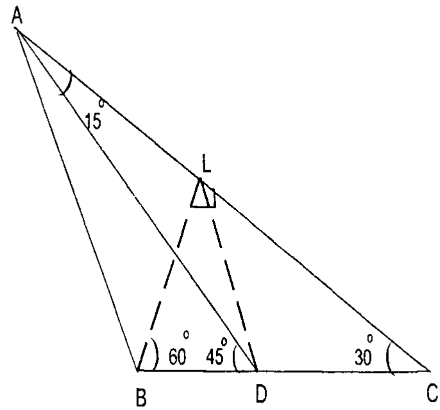
Let the diagonal AC and BD meet at W.

Then $DW = \dfrac{DB}{2} = QR = CS = AR$. Thus in $\triangle ADC$, the medians AR, DW, CS are all equal. Thus $\triangle ADC$ is equilateral.

This implies that $ABCD$ is a rhombus.

Moreover the angles are $60°$ and $120°$.

3.69: Draw BL perpendicular to AC; join L to D.



Since $\angle BCL = 30°$, we get $\angle CBL = 60°$ Since BLC is a right angled triangle with $\angle BCL = 30°$.

We have $BL = BC/2 = BD$.

Thus in $\triangle BLD$, we have $BL = BD$ and $\angle DBL = 60°$.

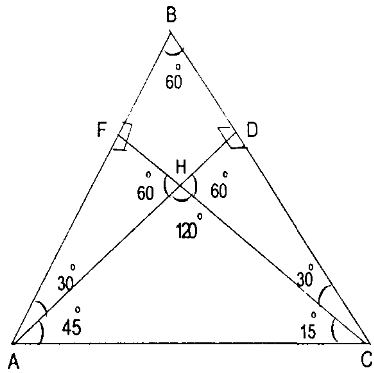
$ADB = 45°$; $\therefore$ We get $\angle ADL = 15°$.

But $\angle DAL = 15°$. Thus $LD = LA$. We thus have $LD = LA = LB$.

This implies that L is the circumcentre of $\triangle BDA$.

$$\therefore \angle BAD = \frac{1}{2}\angle BLD = \frac{1}{2} \times 60° = 30°.$$

3.70: Now $\angle FHD = 180 - B = 120°$. As B, F, H, D are conycyclic.



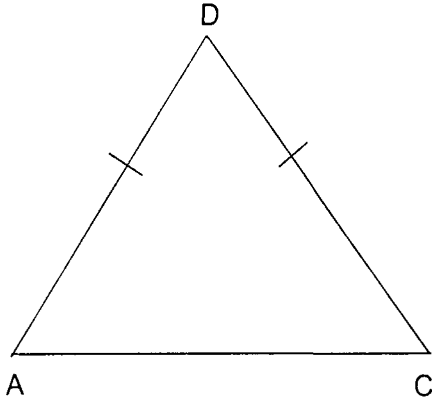
$\therefore \angle DHC = 60°, \angle FHA = 60°$ (linear pair with FHD)

$$\angle AHC = 120°(\text{ver.oppo} < s)$$

$$\therefore \angle FAH = \angle HCD = 30°(D \text{ property })$$

$$\therefore \angle HAC = 75° - 30° = 45° \text{and}$$

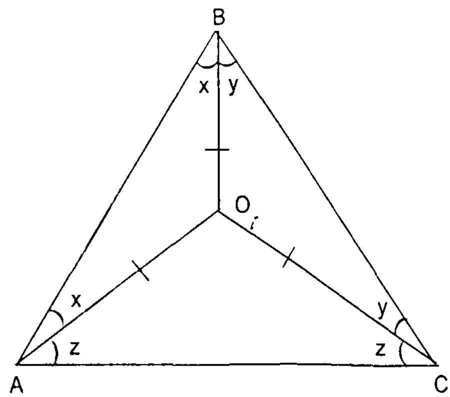
$$\angle HCA = 45° - 30° = 15°$$



$\triangle ADC$ is isosceles with $AD = CD$.

$\therefore \angle A = \angle C = 45°$ in $\triangle ADC$.

$\therefore$ Perpendicular bisector of AC passes through circumcentre O. But it is also angular bisector of $\angle D$. $\therefore \bar{DO}$ bisects $\angle ADC$ or $\angle HDC$.

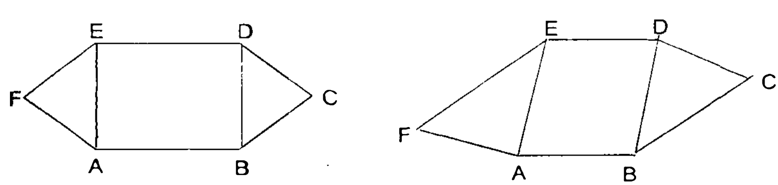


Now

$$x + y = 60$$
$$y + z = 45$$
$$z + x = 75 \Rightarrow x = 45°$$
$$y = 15°$$
$$z = 30°$$

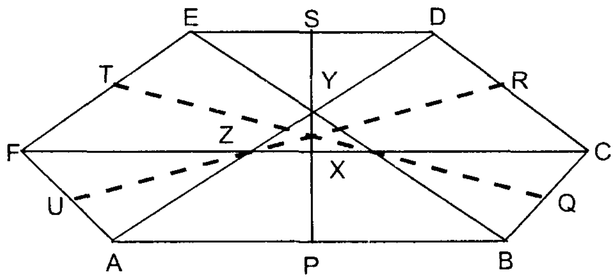
$\therefore \overline{CO}$ bisects $\angle HCD$. Hence O is the incentre of ΔHCD.

3.71: Let the opposite sides be unequal.



For, if $AB = DB$, then $ABDE$ is a parallelogram . The only parallelograms which are concyclic are the rectangles.

Assume now that in the given hexagon $ABCDEF$, the opposite sides are unequal and all the six conditions hold good. Let P, Q, R, S, T and U be the mid points of the sides AB, BC, CD, EF and FA respectively.



Case1: AD, BE, CF meet at a point O. We have $ABDE, BCEF$ and $CDFA$ as isosceles trapezium.

Hence O must be on the perpendicular bisectors of AB, BC, CD, DE, EF and FA. This means that $OA = OB = OC = OD = OE = OF$ and $ABCDEF$ is a cyclic hexagon with O as its centre.

Case(2): Assume that AD, BE, CF are not concurrent. then they from a triangle XYZ as shown in figure now PX extended, RY extended and TZ extended should be the internal angle bisectors of $\triangle XYZ$. Therefore they meet at the incentre I of $\triangle XYZ$. I lies on the perpendicular bisector of each of the sides of the hexagon. Thus A, B, C, D, E, F lie on a circle with centre I.

Second Part: we are going to prove that any five of the six statements are enough that the hexagon is cyclic.

Suppose that

$$AB \parallel DE, AE = BD, BC \parallel EF.$$

$$BF = CF \text{ and } CE \parallel CD \text{ and } CD = AF$$

Then we have,

$$AD = BE = CF.$$

Now

$$\Delta YCD \sim \Delta YFA.$$

$$\therefore \frac{FY}{AY} = \frac{CY}{YD} = \frac{FY + CY}{AY + YD} = \frac{CF}{AD} = 1.$$

$\therefore FY = AY$ and $CY = YD$.

This implies that $\Delta CYA \equiv \Delta DYF \therefore AC = DF$.

This means that all the six conditions hold good and the hexagon is cyclic.

Suppose that $AB \parallel DE, AE = BD, BC \parallel EF$.

$BF = CE$ and $AC = DF$. Now we shall thus that the sixth condition, viz $CD \parallel FA$ is also satisfied.

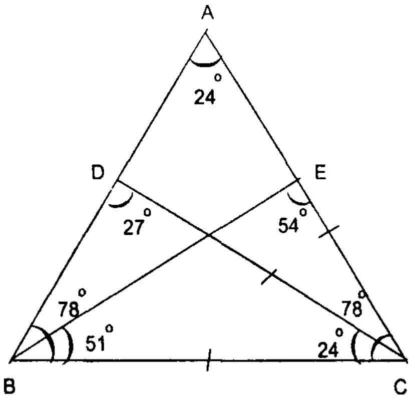
Now $AD = BE = CF \therefore \Delta FDC = \Delta ACD$.

This means that $\angle ADC = \angle FCD$.

Similarly $\angle CFA = \angle DAF$.

These observations imply that $CD \parallel FA$. Thus, whenever any five of the six statements ar true, then the sixth statement is also true and the hexagon is cyclic.

3.72: Now $\angle CDB = \angle DCA + \angle CAD = 78^\circ$ as $\angle A = 24^\circ$.



$\angle BEC = 51°$. Thus $BC = CD = CE$.
(Since $\triangle DBC$ and $\triangle BCE$ are isosceles).

Thus C is the centre of the a circle that passes through B, D, E with radius equal to BC.

$\therefore \angle DEB = \frac{1}{2}\angle DCB = \frac{1}{2} \times 24° = 12°$
(central angle theorem).

3.73: Extend BC to E much that $CD = CE$ and join DE. Now $\angle BAC = 20°$ and $AB = AC$.

$\therefore \angle ABC = \angle ACB = 80°$.

Also $\angle ADC = 100$ and $DA = DC$

$\therefore \angle DAC = \angle DCA = 40°$

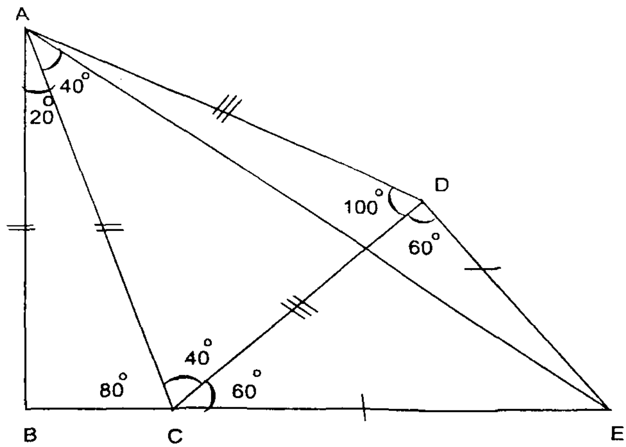
As $\angle ACB = 80$ and $\angle DCA = 40, \angle DCE = 60°$

Also $CD = CE \therefore \triangle CDE$ is equilateral.

$\therefore \angle ADE = \angle ADC + \angle CDE = 100 + 60 = 160°$

Also $AD = DC = CE$.

$\therefore \angle DAE = \angle DEA = 10°$



So $\angle BAE = \angle BAD - \angle DAE = 60 - 10 = 50°$
and $\angle AEB = \angle DEB - \angle DEA = 60 - 10 = 50°$

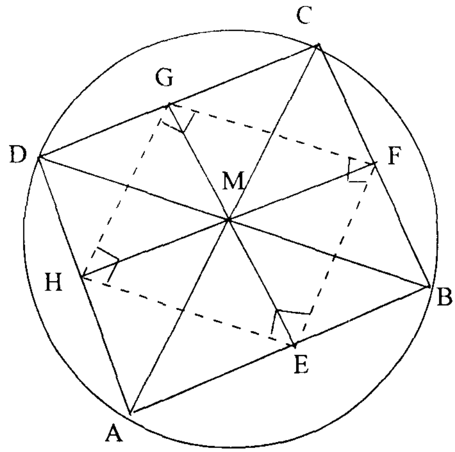
$\therefore$ In $\Delta BAE, \angle BEA = 50$. i.e., $BE = BA$.

Hence $AB = BE = BC + CE = BC + CD$

3.74: $ABCD$ is a cyclic quadrilateral. Also ME, MF, MG and MH are perpendiculars in AB, BC, CD, DA respectively. So quadrilaterals $AEMH$, $BFME$, $CGMF, DHMG$ are all cyclic.

In a cyclic quadrilateral, angles subtended by a side at different points of one of the arcs (major of mirror) are all equal. Therefore, considering the quadrilaterals $ABCD$, $AEMH$, $BFME$.

We have, $\angle HEM = \angle HAM = \angle DAC = \angle DBC = \angle MBF = \angle MEF$. So EM is the internal bisector of $\angle HEF$.



Similarly by choosing appropriate quadrilaterals, we can show that MF, MG, MH are internal bisectors of the angles, $\angle EFG, \angle MGH$ and $\angle GHE$ respectively.

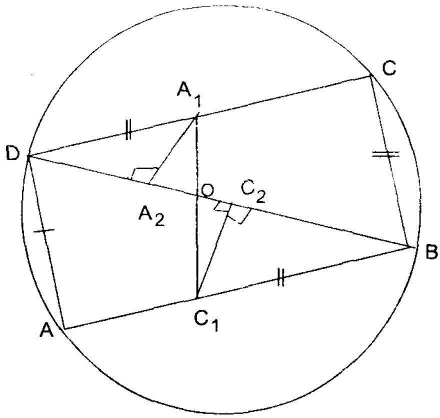
Hence M is a point on the internal bisectors of angles of the quadrilateral, the perpendicular distance of M from the sides are equal.

So there is a circle with centre M touching all the sides.

3.75: Suppose O is the point of intersection of BD with A_1C_1. Draw $A_1A_2, C_2C_2 \perp r$ to BD.

$$\begin{aligned}\therefore A_1A_2 &= DA_1 \sin\angle A_1DA_2 \\ &= DA \sin\angle A_1DA_2 \\ &= 2R\sin\angle C_1BC_2 \sin\angle A_1DA_2 \qquad (A)\end{aligned}$$

(in any $\triangle ABC, A = 2R\sin A$ etc).



$$\begin{aligned} C_1C_2 &= BC_1 \sin \angle C_1BC_2 \\ &= BC \sin \angle C_1BC_2 \\ &= 2R(\sin \angle A_1DA_2) \sin \angle C_1BC_2 \qquad (B) \end{aligned}$$

$A_1A_2 = C_1C_2$ (from (A) and (B)).

If A_1C_1 is $\perp r$ to BD, then A_2 and C_2 coincide with the midpoint of A_1C_1.

Otherwise the triangles A_1OA_2 and C_1OC_2 are congruent and hence $A_1O = C_1O$ as required.

★★★★★

UNIT 3: NUMBER SYSTEM

4.01: Writing the number as

$$\frac{1}{10+3\sqrt{11}}, -\frac{1}{10+3\sqrt{11}}, -\frac{1}{18+5\sqrt{13}},$$

$$\frac{1}{51+10\sqrt{26}}, -\frac{1}{51+10\sqrt{26}},$$ we see that

$$\frac{1}{51+10\sqrt{26}}$$ is the smallest positive number

4.02: $$\frac{2\sqrt{6}}{(\sqrt{2}+\sqrt{3}+\sqrt{5})} = \frac{2\sqrt{6}(\sqrt{2}+\sqrt{3}-\sqrt{5})}{(\sqrt{2}+\sqrt{3})^2-5}$$

$$= \frac{2\sqrt{6}(\sqrt{2}+\sqrt{3}-\sqrt{5})}{5+2\sqrt{6}-5} = \sqrt{2}+\sqrt{3}-\sqrt{5}.$$

4.03: $n^{200} < 5^{300} \Rightarrow n^{200} < (5^{\frac{3}{2}})^{200} \Rightarrow n < 5^{\frac{3}{2}}$

$= (\sqrt{5})^3 = 5\sqrt{5}$.

To find $n < 5\sqrt{5} \lesssim n+1 \Leftrightarrow n^2 \leq 125 \leq (n+1)^2$

$\therefore n = 11$ by inspection.

II Method: $n^{200} < 5^{300} \Rightarrow (n^2)^{100} < (5^3)^{100}$

$\Rightarrow n^2 < 5^3$ i.e., $n^2 < 125$.

$\therefore$ Smallest integer such that $n^2 < 125$ is $n = 11$.

4.04:

$$\sqrt{3+2\sqrt{2}} - \sqrt{3-2\sqrt{2}} = \sqrt{(\sqrt{2}+1)^2} - \sqrt{(\sqrt{2}-1)^2}$$

$$= (\sqrt{2}+1) - (\sqrt{2}-1) = 2$$

which is an integer.

Note: $\sqrt{(\sqrt{5}-1)^2} = \sqrt{2}-1$ since radical always denotes the non-negative square root.

4.05: By Pythagoras Theorem, $(a+2d)^2 = a^2 + (a+d)^2$.

$$\Rightarrow a^2 - 2ad - 3d^2 = 0 \Rightarrow (a-3d)(a+d) = 0$$

$\therefore a = 3d$ or $a = -d$; But $a = -d$ is not admissible as a and d are positive.

$$\therefore \frac{a}{d} = 3 \text{ or } a : d = 3 : 1.$$

4.06:

$$\frac{\sqrt{\sqrt{5}+2} + \sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}} = \frac{\sqrt{\sqrt{5}+2+\sqrt{5}-2+2}}{\sqrt{\sqrt{5}+1}}$$

$$= \frac{\sqrt{2\sqrt{5}+2}}{\sqrt{\sqrt{5}+1}} = \sqrt{2}\frac{\sqrt{\sqrt{5}+1}}{\sqrt{\sqrt{5}+1}} = \sqrt{2}.$$

Thus

$$\begin{aligned} N &= \sqrt{2} - \sqrt{3-2\sqrt{2}} \\ &= \sqrt{2} - \sqrt{(\sqrt{2}-1)^2} \\ &= \sqrt{2} - (\sqrt{2}-1) = 1 \end{aligned}$$

which is a positive integer.

4.07: $1260 = 3^2 \times 7 \times 2^2 \times 5$.

To make it a perfect cube, we have to multiply it by $3 \times \times 7^2 \times 2 \times 5^2 = 7350$ at least.

$$\therefore x = 7350$$

4.08: If the sum of the digits is odd, the only possibilities are

(i) 2 digits even, one digit odd

(ii) all the three digits odd.

In case(i), the odd digit can be any one of $1, 3, 5, 7, 9$. the even digits can be anyone of $0, 2, 4, 6, 8$ where 0 cannot occur in the hundredth place.

If the hundredth place has an odd digit, three are 5 choices, For such choice, the tenth place has 5 choices and the unit place has 5 choices. Thus there are $5 \times 5 \times 5 = 125$ such numbers. (A)

If the tenth or unit place has an odd digit, the hundredth place has 4 choices (excluding 0) and the other place has 5 choices. In each such case, we have therefore $5 \times 4 \times 5 = 100$ choices. (B)

In case (i), we have therefore $125 + 100 + 100 = 325$ numbers. (C)

In case (ii), each place can be filled in 5 ways so that we have $5 \times 5 \times 5 = 125$ numbers. (D)

Thus the total number of numbers required

$$= 325 + 125 = 450 \qquad \text{(E)}$$

4.09: $72 = 8 \times 9$

8 and 9 are co-prime to each other. So it is sufficient to use the fact that the given number is divisible by 8,9 separately.

It is divisible by 8; then $79b$ is divisible by 8 this will

happen if, $b = 2$ since $0 \le b \le 9$.

If the given number is divisible by 9, then. $a+6+7+9+2$ is divisible by 9 i.e., $(24 + a)$ is divisible by 9.

Since $0 < a \le 9, a = 3$. $\therefore a = 3, b = 2$ is the solution.

4.10: Last two digits of 7^1 are $(0, 7)$,

of 7^2 are $(4, 9)$,

of 7^3 are $(4, 3)$,

of 7^4 are $(0, 1)$,

of 7^5 are $(0, 7)$ etc.

In other words, the last two digits occur in a cycle of 4 i.e., 7^n has two digits.

$$(0,7) \text{ if } n \equiv 1(\text{mod}4)$$

$$(4,9) \text{ if } n \equiv 2(\text{mod}4)$$

$$(4,3) \text{ if } n \equiv 3(\text{mod}4)$$

$$(0,1) \text{ if } n \equiv 0(\text{mod}4)$$

Thus 7^7 has last two digits $(4, 3)$.

$\therefore \quad 7^{7^7}$ has last two digits $(4, 3)$ and for the same reason $7^{7^{7^7}}$ has last two digits $(4, 3)$.

Thus expression wanted is $(4, 3) - (4, 3) + (4, 3) - (0, 7) = (3, 6)$.

4.11: Possible remainders when p_1^2 is divisible by 6 are 1 and 5 only as p_i is a prime.

$\therefore p_i^2$ is of the from $6x + 1$ or $6x - 1$ so should be p_i.

$\therefore p_i^2 \cdots + p_x^2$ is divisible by 6 only when x is divisible by 6.

4.12: Let the digits be a, b, c, d, e; suppose that $a+e < 10$. If $a+e$ is even, there is nothing to prove. Suppose $a+e$ is odd. If $b+d$ is also less than 10, then the middle digit of the sum $M+N$ is 2(mod10) and hence is even. If $b+d = 0$, clearly digit from the right in $M+N = 0$; suppose $b+d > 10$. The number represented by the first two digits from the left in $M+N$ is $a+e+1$ and is even since $a+e$ is odd. Now, let $a+e > 10$; Now $b+d$ since, otherwise the third digit from the right in $M+N$ is $2c$(mod10) and is even. If $b+d$ is odd, the second digit from the right in $M+N$ is $b+d+1$ mod 10 and is even.Same is the case with $b+d$ being even.

4.13: $x^9 = 9^{9^{9^9}}$. $\therefore x = (9^{9^9})^{\frac{1}{9}} = 9^{\frac{9^9}{9}} = 9^{9^8}$

4.14: When decimal number is converted to base x, we keep on dividing by n till the number is no more divisible and the corresponding remainders form the digits of the number in base n. When $40!$ is divided by 13 this way, 0occurs as remainder for the first three times since there are 3 factors of 13 in $40!$ namely $13, 26, 39$.

The number of zeroes with which the resulting number ends is 3.

4.15: Let $p = 1999! + x$ for some $x, 1 < x < 1999$ and $x \in N$. Since $1999!$ is the product of natural numbers from 1 to $1999, x$ is a factor of $1999!$ and for some y we have $p = 1999! + x = x(y+1)$ where y is the product of all numbers from 1 to 1999 except x. This implies that p

has factors other than p and 1.

Hence there exists no prime satisfying the given condition namely,

$1999! + 1 < p < 1999! + 1999$

4.16: It can be directly verified that

$\frac{19971996}{19981997} \neq \frac{1996}{1997}$ and $\frac{19971996}{19981997} \neq \frac{996}{997}$

e.g.,$19971996 \times 1997 = 19971996 \times (1996 + 1)$

$= 19971996 \times 1996 + 19971996 \neq$

$(19971996 + 10001) \times 1996 = 19981997 \times 1996$

$\therefore D$ is different from the rest.

4.17: Every multiple of 5 adds one zero while powers of 5 add x zeroes where x is the index of the power since 5 when multiplied by an even number ends in 0. Hence $5, 10, 15$ add 3 zeroes if multiplied together and then by three even numbers give number divisible by 1000. (and for that we need to multiply all numbers from 1 to 15 to get the smallest number divisible by 10^3). So, the smallest number n such that $x!$ is divisible by 1000 is 15.

4.18: The number of 2-digit numbers that leave remainder 0 when divided by 5 is 18, namely $10, 15, \cdots 95$ and hence those that because 1 as remainder are $11, 16, \cdots 96$. The sum of these is $11 + 16 + 21 + \cdots + 96 = 963$.

Now $963 = 3 \times 3 \times 107$.

So the divisors of 963 are $3, 9, 107, 321, 963$ and 1. Hence the number of divisors is 6.

4.19: 1000^{20} numbers of 60 zeroes.

```
1000 ..... 00 (60 zeroes)   (subtract)
              -20
-------------
99 ..... 9980
```

Hence $1000^{20} - 20$ consists of 58 times and one 8 and one 0. $\therefore$ Sum of the digits $= (58 \times 9) + 8 = 530$.

4.20: $6'$ ends in $6, 6^2 \to 36, 6^3 \to 16, 6^4 \to 96, 6^5$ ends in $76, 66$ ends in $56, 6^6$ ends in 36 and the pattern is repeated $\therefore 6^9$ ends in 96.

4.21:
$$\begin{aligned} n^2 &\equiv \text{the sum of the digits(mod3)} \\ &\equiv 2000(\text{mod}3) \\ &\equiv (2+0+0+0)(\text{mod}3) \\ &\equiv 2(\text{mod}3). \end{aligned}$$

But the square of any integer is always congruent either 0 or 1 modulo 3. So, there is no integer such that the sum of its digits is 2000.

4.22:
$$1999 = (3 \times 666) + 1$$
$$4^{1999} = 4^{3\times 666+1} = (4^3)^{666} \times 4$$
$$= (64)^{666} \times 4 = (6+4)^{666} \times 4 \equiv 4(\text{mod}9).$$

Similarly, $7^{1999} = (7^3)^{666} \times 7 = (343)^{666} \times 7$
$$\equiv 1^{66} \times 7 \equiv 7(\text{mod}9).$$

Hence $4^{1999} + 7^{1999} - 2 \equiv 4 + 7 - 2 \equiv 9 \equiv 0(\text{mod}9)$.

i.e., $(4^{1999} + 7^{1999} - 2)$ is divisible by 9.

4.23: The set is made up of 30consecutive integers. so every alternative integer is either even or odd. So there are 15 even integers and 15 odd integers. We will denote the 15 odd integers in the set (which are alternative integers) by $n_1, n_2 \cdots n_{15}$, in ascending order. Now in the set n_1, n_2, n_3, precisely one integer will be divisible by 3. So, in each of the sets $\{n_1, n_2, n_3\}, \{n_4, n_5, n_6\}$, $\{n_7, n_8, n_9\}, \{n_{10}, n_{11}, n_{12}\}, \{n_{13}, n_{14}, n_{15}\}$, we will find 5 distinct odd integers divisible by 3.

Also in the set $\{n+1, n+2, \cdots n+30\}$, we can definitely find 3 integers ending with $10k+5$, $10k+15$ and $10k+\ 25(k>0$ an integer) which are congruent to $2,0,1$ modulo 3; so one of these three integers is divisible by 5 but not by 2 or 3. So, we have identified in the given set (i) 15 even integers (ii) 5 odd integers divisible by 3. (all these are compose if $n+1>5$) and (iii) 2 odd integers divisible by 5 but not by 3 i.e., we have found $15+5+2=22$ distinct composite numbers. hence the given set has at most eight primes.

4.24: Let the required number be $a \times 10^3 + b \times 10^2 + c \times 10 + d$.

After performing the operations as stated in the problem,

$(a \times 10^3 + b \times 10^2 + c \times 10 + d) - (a \times 10^2 + b \times 10 + c) + (a \times 10 + b) + a = 1999$.

i.e., $a(10^3 - 10^2 + 10 + 1) + b(10^2 - 10 + 1) + c(10 - 1) + d = 1999$

i.e.,$911a + 91b + 9c + d = 1999$. (A)

$$\begin{aligned} \text{Now } 1999 &= 2 \times 911 + 177 \\ 177 &= 1 \times 91 + 86 \\ 86 &= 9 \times 9 + 5. \end{aligned} \qquad (B)$$

The calculations on (B) give the multiples of $911, 91, 9$ and unit contained in 1999 so as to satisfy the condition (A).

(B) gives $a = 2; b = 1; c = 9; d = 5$.

Hence the required number is 2195.

4.25: $\dfrac{n^3-1}{5} = \dfrac{(n-1)(n^2+n+1)}{5}$ is a prime number.

So $(n-1)(n^2+n+1)$ must be divisible by 5.

If $n = 0, 1, 2, 3, 4 \pmod 5$, then $n^2 + n + 1 \equiv 1, 3, 2, 3, 1 \pmod 5$ respectively. Hence $n^2 + n + 1$ is not divisible by 5. Thus we must have $(n-1) = 5k$ where $k \geq 1$. If $k = 1$, then $n = 6$ and $\frac{n^3-1}{5} = 43$, prime. If $k \geq 2$, then $\frac{n^3-1}{5} = k(n^2+n+1)$ is composite. Hence there is only one value of n; i.e., 6.

4.26: $10^{999} = 2^{999} \times 5^{999}$ where 2 and 5 are positive primes. So the number of positive divisors of 10^{999} is $(999+1)(999+1) = 1000^2$. (A)

Similarly the number of positive divisors of 10^{998} is 999^2. (B)

So, the number of positive numbers which divide.

$$10^{999} \text{ but not } 10^{998} \text{ is } 1000^2 - 999^2 = 1999.$$

4.27: The given number is $1234\cdots979899100$. The number of times the digit 1 occurs is

in $1-9$ - once

in 10 - once

in $11-19$ - ten times

in 20 to 99 - eight times

and in 100 - once

hence the total number of times 1 occurs is 21.

4.28: The given sequence of numbers is

$$1, 23, 45, 67, 89, 1011, 1213\cdots 9899, 100101, 102\ 103.$$

The four digit numbers are $1011, 1213, 1415\cdots 98999$.

These form an AP with first term 1011 and common difference 202. Number of terms is

$$\frac{9899-1011}{202}+1=45.$$

4.29: As $n!$ has four zeroes at the end, it should have 5^4 as a factor. hence x can be $20, 21, 22, 23$ or 24. Further, $(n+1)!$ must have 6 zeroes at the end. This is possible if $n+1=25$ or $n=24$.

4.30: As $x^2-y^2=(x+y)(x-y)=1$, we have either $x+y=1, x-y=1$ or $x+y=-1, x-y=-1$. The integer solutions (x,y) are given by $(1,0)$ and $(-1,0)$.

4.31: 3 does not divide $x_i, i=1,2,\cdots 10$. Then x_1^2 leaves

remainder 1 on divisors by 3 for

$$\text{if } x_i = 3k+1,\ x_i^2 = (3k+1)^2 = 9k^2+6k+1 = 3(3k^2+2k)+1$$

and if

$$x_i = 3k+2, x_i^2 = (9k+2)^2 = 9k^2+12k+4 = 3(3k^2+4k+1)+1.$$

$$x_1^2 + x_2^2 + \cdots + x_{10}^2 = \text{a multiple of } 3+10 = \text{a multiple of} 3+1.$$

Hence this sum leaves remainder 1 when divided by 3.

4.32: $a^3 - b^3 - c^3 = 3abc$

$\Rightarrow a^3 + (-b)^3 + (-c)^3 = 3abc$

$\Rightarrow a - b - c = 0 \ \Rightarrow a = b + c.$ (A)

But $a^2 = 2(b+c)$.

This implies $(b+c)^2 = 2(b+c)$.

i.e., $b + c = 2$ or $b = -c$

$\therefore a = 2$ or $a = 0$. (B)

As a is a natural number, $a \neq 0; \therefore a = 2$.

4.33: $2001 = 3 \times 23 \times 29$.

As $2001 = 3 \times 23 \times 29$ and the numbers used are even, the product must include $6, 46, 58$. Hence the largest divisor which is even, which is used, must be 58.

4.34: The last two digits of the square root is of the form $x4$ or $x6$. Hence the number is

$$M(10) + 10x + 4 \text{ or } M(10) + 10x + 6. \tag{A}$$

Hence its square is an even multiple of 10 plus 16 or an even multiple of 10 plus 36. Hence the ten's digit of the number is an even number plus 1 or 3. (B)

In any case, the ten's digit is an odd number.

4.35: The single funny numbers are $2, 3, 5, 7$. The 2-digit funny numbers are $23, 37, 53, 73$. (Since the number itself must be prime and the digits also must be primes).

In general, a K-digit number is funny implies that each digit is funny; every m-digit part of the number is also funny for $1 \leq m \leq k$. This means that only 3 -digit funny numbers are the prime numbers from $237, 373, 537, 737$ (note that the last digit cannot be 2 or 5). Thus the only 3 digit funny number is 373. Again, this means that there are no 4-digit it higher digit funny numbers. Thus the set of all funny numbers is

$$\{2, 3, 5, 7, 23, 37, 53, 73, 373\}.$$

4.36: Let the prime numbers be p and q with $p = q+100$. Since 100 is 1 mod 3 and q is a prime we must have $p \neq 2\text{mod}3$. If $p = 1\text{mod}3$ and the concatenated numbers pq (it is p Justaposed to q and not product p with q) and qp both become multiples of 3. So, neither of them is a prime. If one of the concatenated numbers were to be prime number, then, $p \neq 1$ mod3.

Hence p must be 3 and $q = 103$.

Now 3103 is a multiple of 29 and so it is not a prime. the

number 1033 as a prime.

$$\therefore 3, 103, 1033 \text{ are the primes required.}$$

4.37: Let the numbers be $n+1, n+2, x+3, \cdots, n+k$ (with $k \geq 2$) and $(n+m)$ be the number removed where $1 \leq m \leq k$.

Then we have,

$$\frac{(n+1)(n+2)+\cdots+(n+k)-(n+m)}{k-1} = 50.55$$

$$\therefore 2n(k-1)+k(k+1)-2m = (101.1)(k-1). \qquad \text{(A)}$$

In the above equation (A), L.H.S is an even integer and therefore $(k-1)$ must be multiple of 20. (B)

Let $k = 20t+1$. (C)

$$\begin{aligned} 2m &= 2n(20t)+(20t+1)(20t+2)-(101.1)20t \\ &= 40nt+400t^2+60+2-2022t \\ &= 40nt+400t^2-1962+2. \end{aligned}$$

$\therefore m = 200t^2+(20n-98)t+1 \leq k = 20t+1$

$\therefore t(200t+20n-1001) \leq 0$ (since $k \geq 2 \Rightarrow t \neq 0$)

$\therefore 200t \leq 1001-20n \leq 1001$

$\therefore t \in \{1,2,3,4,5\}$.

Now $t = 1$ gives $k = 21$ and $x = 20n - 780$. Also $1 \leq n \leq k$ gives $1 \leq 20n - 780 \leq 21$. So $n = 40$ and the required set is $\{41, 42, 43, \cdots 61\}$ and the removed number is 60. Similarly we work out for $t = 2, 3, 4, 5$. We get for $t = 2$, the set is $\{31, 32, 33, \cdots 71\}$ and the number removed is 69.

For $t = 3$, the set is $\{21, 22, 23, \cdots 81\}$ and the number removed is 78.

For $t = 4$, the set is $\{11, 12, 13, \cdots 91\}$ and the number removed is 87.

For $t = 5$, the set is $\{1, 2, 3, \cdots 101\}$ and the number removed is 96.

4.38: Let X be the total number of medals awarded. We construct the following table.

Day	Number of medals awarded	Number of medals remaining
1	$1+\frac{X}{2}$	$X-(1+\frac{X}{2}) = \frac{X-2}{2}$
2	$1+\frac{(X-2)}{4}$	$\frac{X-6}{4}$
3	$1+\frac{X-6}{8}$	$\frac{X-14}{8}$
4	$1+\frac{X-14}{16}$	0

As medals awarded on the fourth day is equal to the balance of medals at the end of the third day

$$\frac{X-14}{16} + 1 = \frac{X-14}{8}$$

$$\text{i.e., } \frac{x+2}{16} = \frac{2 \times -28}{16}$$

$$\therefore X = 30.$$

$\therefore$ The number of medals $= 30$.

Medals awarded on successive days are $16, 8, 4, 2$.

4.39: Let 2^{2001} have p digits when written in full.

5^{2001} have q digits when written in full

$$\therefore 10^{p-1} < 2^{2001} < 10^{p}$$

$$\therefore 10^{q-1} < 5^{2001} < 10^{q}$$

$$\therefore 10^{p+q-2} < 10^{2001} < 10^{p+q}$$

$$\therefore 2001 = p + q - 1 \text{ or } p + q = 2002.$$

4.40: The last two digits will have to be even 25 or 75.

4.41: Suppose there are two numbers n and m in A such that m divides n (m/n). Let $n = mk$.

Then we must have $2 \leq k \leq \dfrac{7654321}{1234567} < 7$ (A)

This means that $k \in \{2, 3, 4, 5, 6\}$. (B)

Now $1+2+3+4+5+6+7 = 28$ and therefore no element in A is divisible by 3. This implies K is not 3 or 6 (C)

Let $k = 2$: Here $n = 2m$ and $n + m = 3m$. But both m and n are 1 mod 3 since each element of A has sum of its digits equal to $28 = 1\text{mod}3$. So $m + n = 2\text{mod}3$ and therefore $k = 2$ is not possible.

Let $k = 5$. In this case $n = 5m$ and $m + n = 6m$. The same reasoning tell us that it is also possible. Let $k = 4$. Here $n = 4m$ and $n - m = 3m$. Now $m = n = 28 = 1\text{mod}9$ implies that $(n-m)$ is a multiple of 9. $\therefore (n - m) = 3m = 9p$ or $m = 3p$ which is impossible.

The above analysis shows that one or two numbers in A such that one of them is divisible by the other.

4.42: $P^2 + 7pq + q^2 = a^2$ for some $a \in N$.

$\therefore a^2 - (p+q)^2 = 5pq$ i.e., $(a+p+q)(a-p-q) = 5pq$ (A)

As $a+p+q>5$ or $p>q$, we have $a+p+q=5p$ or $5q$ or pq. (B)

If $a+p+q=5p$, then $a-p-q=q$.

$\therefore 2(p+q)=5p-q \Rightarrow 2p+2q=5p-q$.

$\Rightarrow 3p=3q \quad \Rightarrow p=q$.

The case $a+p+q=5q$ is similar. (C)

If $a+p+q=pq$, then $a-p-q=5$ and so $2p+2q=(a+p+q)-(a-p-q)=pq-5$ and so $pq-2p-2q+4=5+4=9$, i.e., $(p-q)(q-2)=q=1\times 9=9\times 1=3\times 3$. Taking all positive cases, $(p-2,q-2)=(1,9)$ or (9,1) or (3,3).

$$\text{i.e., } (p-q)=(3,11) \text{ or } (11,3),(5,5).$$

Of these the last pair $(5,5)$ is already included in (B). Hence the solution set (p,q) is

$$\{(p,p)|p \text{ a prime } \cup 9(3,11),(11,3)\}.$$

4.43: As there are n rows in the array and each row consists of the n numbers $1,2,3,\cdots n$, there is a total of n^2 numbers and each of the numbers $1,2,\cdots n$ occurs n times in the array. Also as $a_{ij}=a_{ji}$ for the $1\leq i,\ j\leq n$ $(i,J\in N)$ and n, is odd $(=2k+1)$ say; array is symmetric about the diagonal.

$$D=\{a_{11}\ a_{22}\ a_{33}\cdots a_{nn}\}$$

of the array. So the number of times any specific number of the set $S=\{1,2,3,\cdots n\}$ will occur on both sides of the diagonal but not on D is even (say $2m,m\leq k,m\in N$). So this specific number will occur an odd number of times

($= 2k+1-2m$) in D. This holds for each of the numbers $1, 2, 3, \cdots n$ i.e., each number of $1, 2, 3 \cdots n$ occurs at least once in D.

Now D has n numbers $a_{11}, a_{22}, a_{33}, \cdots a_{nn}$, as its elements and also each of the numbers $1, 2, 3, \cdots n$ occurs at least once in D. Hence the numbers $a_{11}, a_{22}, a_{33}, \cdots a_{nn}$ are the numbers $1, 2, 3 \cdots n$ in some order.

4.44: Let $2^{n+1} - 1 = P$. Then we are given that p is a prime number. We have $2^{n+1} = p+1$ and $2^n = \frac{P+1}{2}$. (A)

This makes $N = 2^n \times P$. We find that the divisors of $2^n \times P$ are $1, 2, 2^2, 2^3 \cdots 2^n, P, 2P, 2^2P, 2^3P \cdots 2^nP$ So, the sum of the divisors of N is given by

$$\begin{aligned}
\text{(i) } \sigma_N &= 1 + 2 + 2^2 + 2^3 \\
&\quad + \cdots + 2^n + P(1 + 2 + 2^2 + 2^3 + \cdots + 2^n) \\
&= (P+1)\left(\frac{2^{n+1}-1}{2-1}\right) \\
&= (P+1)(2^{n+1}-1) \\
&= 2N.
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } \sigma_{\frac{1}{N}} &= (P+1)\left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\right] \\
&= (P+1)\left[\frac{1 - \frac{1}{2^{n+1}}}{1 - 1/2}\right] \\
&= (P+1)\left(\frac{2^{n+1}-1}{2n}\right) \\
&= 2^{n+1}\left(\frac{2^{n+1}-1}{2n}\right) = 2(2^{n+1}-1) \\
&= 2P.
\end{aligned}$$

4.45: We observe that any permutation of $\{1,2,3,4,5,6\}$ can be reached in atmost 5 transpositions for any permutation P given by $P_1, P_2, P_3, P_4, P_5, P_6$, we count the number of pairs $i < j$ such that $p_i < p_j$ and call it $c(P)$. Then every transposition multiplied to P changes the $(odd - even)$ parity of $c(P)$. This mans that the set of all permutations can be separated into those reachable by an odd numbers of transpositions and those reachable by an even number of transpositions. Any permutation reachable in 0 or 2 steps can also be reached in four by repeating the same transposition twice. So we are required to count the number of even permutations.

Fix a transposition T. Then T defines a correspondence $P \to TP$ from a set of even permutations to the set of odd permutations, which is clearly one-one. This means that the number of even permutations is one half of the total number of permutations, which is $6! = 720$ in our case.

$\therefore$ The number of permutations is 360.

4.46: Looking at terms of the sequence given, it can be noted that all even numbers have the same number of a's and b's while, all odd numbers have 'a' more than the b's. Further in the n^{th} member, if n is even, the number of a's=the number of b's$=\frac{n}{2}$ and if n is odd, the number of a's $= \frac{n+1}{2}$ while the number of b's is $= \frac{n-1}{2}$.

As 2004^{th} number of the sequence corresponds to an even number 2004, the number of a's and the number of b's in this member will both be equal to $\dfrac{2004}{2} = 1002$.

As 2003 is odd, the number of a's in the 2003^{rd} number 1002 and the number of b's will be $\frac{2003+1}{2} = 1001$.

$\therefore$ Number a's in the first 2004 numbers

$$= 1 + 1 + 2 + 2 + 3 + 3 + \cdots + 1002 + 1002$$

$$= 2(1 + 2 + \cdots + 1002) = \frac{2 \times 1002 \times 1003}{2} = 1002 \times 1003$$

Also number of b's in the first 2004 numbers.

$$0 + 1 + 1 + 2 + 3 + 3 + \cdots + 1001 + 1002$$

$$= 2[1 + 2 + \cdots + 1001] + 1002 = 1002^2.$$

4.47: Let the number be denoted by ab so that it is equal $(10a+b)$. As ab, the product of the digits a and b divides $(10a+b)$, we must have 'a' as a divisor of 'b' (A)

i.e.,$a \leq b$ (B)

As a and b are from $0, 1, 2 \cdots 9$, it follows that 'a' can at best be 4 i.e., $a \leq 4$. (C)

When $a = 1$, of the numbers $11, 12, \cdots, 19$, only $11, 12, 15$ have the defined property.

When $a = 2$, of the numbers $21, 22, \cdots 29$, only 24 has the defined property.

When $a = 3$, of the numbers $31, 32, \cdots 39$, only 36 has the defined property.

When $a = 4$, of the numbers $41, 42, \cdots 49$, none has the defined property.

Thus the number of numbers with the stated property is 5, namely, $11, 12, 15, 24, 36$.

4.48: Since the problem involves nine distinct digits A, B, C, D, G, H, I, J and K, only one of the digits from $0, 1, 2, 3, \cdots 9$ is omitted. (A)

The sum of two digits is another single digit or
a double digit number with possibly a carry over 1. (B)

This implies $G = 1$. (C)

But A together with carry then 10.

$\therefore$ A has to be 9 and the carry is 1 from hundreds place addition. This forces H to be 0. (D)

Then I being distinct from H is not 0.

Thus $B + B$ plus a carry of one from the ten's place addition will be greater than or equal to 11.
Hence $B \geq 5$. (E)

If $B = 5$, then I is either 0 or 1. But already o and 1 are assigned to H and G respectively. Thus B is equal to $6, 7$ or 8 as 9 is already taken by A. The same argument shows that C and D also cannot be equal to 5. The addition of D to itself gives the sum an even number. (F)

In the unit's place i.e., k is even; So $k \neq 5$. (G)

Let us see whether $J = 5$. This is possible only if $C = 2$ or $D > 5$. If $C = 7$, then $B = 6, D = 8$ or $B = 8, C = 6$. If $B = 8$, then $D \neq 8$. (H)

Similarly if $B = 6$, then $K = 2$. But then with a carry,

from ten's place , $I = 7$, a digit taken by C . (I)

This is not possible. Hence $C \neq 7$. Thus when $J = 5, C$ can only be equal to 2. D has three possible values $6, 7, 8$. Corresponding to which B has the values 7 or 8, 6 or 8, 6 or 7 respectively. (J)

A solution exists only when $D = 7$ and $B = 8$; namely $9827 + 827 = 10654$.

The digit 3 alone is missing in this equation.

Note: we have a few other solutions such as $9782 + 782 = 10564$; $9728 + 728 = 10456$ etc.

In all the above, the digit 3 will be missing.

4.49: Let the children have a, b, c, d, e rupees such that they are integers and $a \geq b \geq c \geq d \geq e$. (A)

Now $a + b + c + d + e = 847$ and b, c, d, e are divisors of $a; c, d, e$ are divisors of $b; d, e$ are divisors of $c; e$ is the divisor of d. (B)

As e divides exactly a, b, c, d and e, it is a factor of

$a + b + c + d + e = 847 = 7 \times 11 \times 11$. (C)

In other words, $e = 7$ or 11 or 7×11 or 11×11. (D)

But the last value obviously is 7. (E)

Hence any child must have had at least RS 7. (F)

A possible solution for a, b, c, d will be, $a = b = c = d = 210$ and $e = 7$. (G)

4.50: The number is a multiple of 137 and 73. (A)

As 137 and 73 and primes, the number is also a multiple of $137 \times 73 = 10001$. The eight digit number which is a multiple of 1001 can be got by multiplying a four digit number $abcd$ by 10001. The product is obviously, $abcd\ abcd$. The second and the sixth digits from the left are both equal to b. Hence, if the second digit is 7, then the sixth digit is also 7.

4.51: Let x be the smallest positive integer such that $1260n = n^3$ for a positive inter n.

As $1260 = 2^2 \times 3^2 \times 5 \times 7, 1260n$ will be a cube if

$n = 2 \times 3 \times 5^2 \times 7^2 = 7350$.

For this value of $n, 1260 \times 7350 = (2 \times 3 \times 5 \times 7)^3 = (210)^3$.

Thus $1000 < n < 10000$.

4.52: The number 43 to base x is equal to the decimal $4x + 3$. Similarly the number 34 to base y is equal to the decimal $3y + d$. As these two decimal numbers are equal.

$4x + 3 = 3y + 4$ or $4(x + y) = 7y + 1$.

In other words $4(x+y) - 1$ is a multiple of 7. If $x+y = 9$, then $4(x + y) - 1$ is a multiple of 7.

Also if $x + y = 16$, then $4(x + y) = 1$ is a multiple of 7. $\therefore x + y = 16$ is the solution. Why?

If $x+y = 9$, x or y can be 4 or 3 etc which is not allowed.

4.53: The problem is one of finding the number of fractions $\frac{a}{b}$ of the given set such that $a < 1002$ and H.C.F of a, b is 1.

Now $2004 = 2^2 \times 3 \times 167$. As $(a+b)$ is even and a, b do not have a common factor other than 1, both a and b are odd. when a is a multiple of 3, since 2004 is a multiple of $3, b$ is also a multiple of 3 and H.C.F $(a, b) \neq 1$. Similarly when a is a multiple of 167 i.e., when $a = 167, 334, 501, 668$ or 835 takes the values $11 \times 167, 10 \times 167, 9 \times 167, 8 \times 167$ or 7×167, which are multiples of 167.

Hence H.C.F $(a, b) \neq 1$.

Even numbers from 1 to 1001.i.e., Multiples of 2 are 500; multiples of 167 are 5.

And odd multiples of 3 are 164. Thus corresponding to these $500 + 5 + 164 = 669$ values of a, the number b will be such that H.C.F (a, b) is not 1. For the remaining $1001 - 669 = 332$ values of a, in $\{1, 2, \cdots 1001\}$, the conditions of the problem are satisfied. Hence the number of fractions with the defined property in the given set of fractions is 332.

4.54: For $\sqrt{9 - (n+2)^2}$ to be real,

$9 - (n+2)^2 \geq 0$ or $(n+2)^2 \leq 9$ (A)

Hence $n + 2 = -3 - 2, -1, 0, 1, 2, 3$. (B)

As n is an integer, n can be $-5, -4, -3, -2, -1,$ 0 or 1.

Hence there are seven possible values for x. (C)

4.55: We have $(x+1)^2 = x^2 + 2x + 1$ so that the integer parts of

$\sqrt{x^2}, \sqrt{x^2+1}, \sqrt{x^2+2} \cdots \sqrt{x^2+2x}$ will all be x.

$$\therefore\ [\sqrt{x^2}] + [\sqrt{x^2+1}] + \cdots + [\sqrt{x^2+2x}]$$
$$= (2x+1).x = 2x^2 + x\,. \qquad \text{(A)}$$

Also $2004 = 1936 + 68 = 44^2 + 68$

$$\therefore [\sqrt{44^2}] + [\sqrt{44^2+1}] + \cdots + [\sqrt{44^2+68}] = 69 \times 44\,. \quad \text{(B)}$$

Hence the value of the required sum

$$= \sum_1^{43} [\sqrt{x^2}] + [\sqrt{x^2+1}] + \cdots + [\sqrt{x^2+2x}] + 69 + 44\,.$$

$$= \sum_1^{43} (2x^2 + x) + 69 \times 44$$

$$= 2 \times \tfrac{1}{6} \times 43(43+1)(2 \times 43 + 1) + \frac{43 \times 44}{2} + 69 \times 44 \quad \text{(C)}$$

$$= 58,850\,. \qquad \text{(D)}$$

4.56: We can write $7ab73$ as $7ab$ hundreds $+73$.

But this is same as $7ab$ times $99 + (7ab + 73)$. (A)

For 99 to be a divisor of $7ab73$, we therefore have $(7a + b + 73)$ as a multiple of 99. (B)

Giving various digital values to a, b from $0, 1, \cdots 9$ we find that $7ab + 73$ can be a number from $700 + 73$ to $799 + 73$ i.e., a number from 773 to 872. (C)

In this range only 792 is divisible by 99. (D)

$7ab$ coincides with $792 - 73$ i.e., with 719. (E)

$\therefore a = 1; b = 9$.

$\therefore$ There is only one pair of values $(1, 9)$ for (a, b).

Method II: You can also consider $7ab73$ to be divisible by 11 and 9 then analyze to find (a, b).

4.57: If (a, b) where $a > b$ are real numbers, then $(a^n - b^n)$ is divisible by $(a - b)$ for integral $n \geq 1$.

Now $107^{90} - 76^{90} = (107^2)^{45} - (76^2)^{45}$ is divisible by $(107^2 - 76^2)$i.e., by 183 and 31. Now $183 = 3 \times 61$.

Thus the given expression is divisible by 61.

4.58: We have $a_{n+1} = a_n.a_{n-1} + 1 \quad a_0 = a_1 = 1$

$\therefore a_2 = 2; a_3 = 3; a_4 = 7; a_5 = 22; a_6 = 155, a_7 = 3411 \cdots$

Expressed congruent modulo 4, the sequence is $1, 1, \underline{2, 3, 3}, \underline{2, 3, 3}, \cdots$

This is a cyclic sequence from a_2 *onwards,* the numbers $2, 3, 3$ repeating.

$a_{2004} \equiv 3(\text{mod}4)$ 30, 4 does not divide a_{2004}

(If it is a divisor, a_{2004} must be congruent to $0\text{mod}4$).

Similarly, expressing the sequence congruent modulo 2, modulo 3 and modulo 5, we get respectively.

$(i) 1, 1, 0, 1, 1, 0, 1, 1 \cdots$ congruent modulo 2

(ii) $1, 1, 2, 0, 1, 1, 2, 0 \cdots$ congruent modulo 3

(iii) $1, 1, 2, 3, 2, 2, 0, 1, 1, 2, 3, 2, 2.0 \cdots$ congruent modulo 5

$a_{2003} \equiv 0(\text{mod}2) \therefore 2/a_{2003}$

$a_{2003} \equiv 0(\text{mod}3) \therefore 3/a_{2003}$

$a_{2004} \equiv 2(\text{mod}5) \therefore 5/a_{2004}$.

4.59: Note that

$$x^3 - 1 = (x-1)(x^2+x+1)$$

$$x^3 + 1 = (x+1)(x^2-x+1)$$

$$x^2 + x + 1 = (x+1)^2 - (x+1) + 1$$

ven expression =

$$\frac{2-1)(3-1)(4-1)(4^2+1+4)\cdots(1000-1)(100^2+1000+1)}{+1)(2^2+1-2)(3+1)(4+1)\cdots(1000+1)(1000^2-1000+1)}.$$

As $2^2+2+1 = 3^2-3+1, 3^2+3+1 = 4^2-4+1$ etc.

After cancelations, the given expression

$$= \frac{1}{3}.\frac{2}{4}.\frac{3}{5}\cdots\frac{999}{1001}.\frac{1000^2+1000+1}{2^2-2+1}$$

$$= \frac{2(1000000+1000+1)}{1000\times 1001\times 3} = \frac{2}{3}\times\frac{1001001}{1001000}$$

$$= \frac{333667}{500500} = 0.666001332$$

i.e., $\simeq 0.666$.

$\therefore$ The best approximation is $333/500 \simeq 0.666$.

4.60: $10^{(10^{10})} = 10^{(10^8\times 10^2)} = (10^{10^8})^{100}$

$\therefore\ \sqrt[100]{10^{10^{10}}} = 10^{10^8}$.

4.61: Let the three consecutive natural numbers be $(x-1), x, (x+1)$ respectively.

Consider $(x-1)^3 + x^3 + (x+1)^3$

$$\begin{aligned} &= x^3 - 3x^2 + 3x - 1 + x^3 + x^3 + 3x^2 + 3x + 1 \\ &= 3x^3 + 6x = 3[x^3 + 2x]. \end{aligned}$$

$\therefore\ C = \{3(x^3 + 2x)\backslash x \in N\}$ and $x > 1$ or every element of C is divisible by 3.

4.62: The product of three numbers should be a composite number which can be expressed as product of six primes (need not be distinct but at least two of the primes are distinct).

As $120 = 2^3 \times 3' \times 5', 120$ cannot be expressed as product of six primes.

As $144 = 2^4 \times 3^2, 144$ can be expressed as product of six primes but one has a number that is, not a product of two primes which is not possible.

As $12100 = 2^2 \times 5^2 \times 11^2, 12100$ can be expressed as product of six primes and can be thought as $(2 \times 5) \times (5 \times 11) \times (11 \times 2)$. Three person thought numbers 10,55,22 respectively.

4.63: 5958,5814,5430 leave the same remainder when x divides them,

$\therefore 5958 - 5814 = 144$ is divisible by x

5814-5430=384 is divisible by x

5958-5430=528 is divisible by x.

$\therefore$ The largest possible value is x which will divide the given numbers is the H.C.F of $144, 384, 528$ i.e., 48.

4.64: $\dfrac{1}{5^{2003}} = \dfrac{2^{2003}}{10^{2003}} = \dfrac{2^{4(500)+3}}{10^{2003}}$. As 2^{4x+3} ends with 8 for any natural number x, the last digit of 2^{2003} is 8.

4.65: Let $\dfrac{n}{2003} = d$ and let the last 6 digits of d be $ABCDEF$.

Then we have $d = \cdots ABCDEF, 2003 \times d = n = 5555 \cdots 555.$

It is clear that F must be 5 since $3 \times F$ ends in 5. This means that $E \times 3 + 1$ ends in 5. This forces E to be 8. Thus proceeding this way,

we find that $A = 3, B = 9, C = 5, D = 1, E = 8, F = 5$.

$\therefore$ the last six digits of 'd' are $\cdots 395185$.

4.66: In the unit's place, each of the digits will occur $\dfrac{256}{4} = 64$ times.

In the ten's place, each of the digits will occur in $\dfrac{256}{4} = 64 times.$

In the hundred's place, each of the digits will occur in $\dfrac{256}{4} = 64$ times.

In thousands's place, each of the digits will occur in $\dfrac{256}{4} = 64$ times.

$\therefore$ Total $= 64(1+2+3+4) + (64)(10)(1+2+3+4) + (64)(100)(1+2+3+4) + 64(1000)(1+2+3+4) = 711040$.

4.67: Writing $(2006)^{2005}$ as $(2000+6)^{2005}$, we note that $(2006)^{2005} =$ multiple of $2000 + 6^{2005}$. (A)

Hence the last two digits of this will be the same as the last two digits of 6^{2005}. (B)

Writing n in 6^n as $r(\text{mod}5)$, we find that the last two

digits are $36, 16, 96, 76$ and 56 respectively when $n = 2, 3, 4, 0$ and 1. As $2005 \equiv 0 (\mathrm{mod} 5)$, the last two digits of 6^{2005} are 76.

Hence (2006^{2005}) also has last two digits as 76.

4.68: Enumerating the numbers $2 = 1 + 1, 5 = 1 + 4$,

$13 = 4 + 9, 17 = 1 + 16, 29 = 4 + 25, 37 = 1 + 36$,

$41 = 16 + 25, 53 = 4 + 49, 61 = 25 + 36, 89 = 25 + 64, 97 = 16 + 81$, we find that there are 11 such numbers.

4.69: The number is divisible by 9 and 11.

(i) Sum of the digits is divisible by 9.

(ii) Difference between the sum of the odd positioned digits and the sum of the even positioned digits is a multiple of 11. Thus we have $71 + A + B$ is a multiple of 9 or $(A + B - 1)$ is a multiple of 9 and $(37 + A) \sim (34 + B)$ is divisible by 11.

Thus $A + B = 1, 10$ or 19 and $A - B = -3$.

Clearly $A + B \neq 19$ as A and B are single digits.

If A or B are zero, then the other can be 1 but the difference cannot be 3. The remaining possibility is that $A + B = 10$ and $A - B = -3$. This is also not possible since if $A + B$ is even, then $A - B$ also must be even but $A - B$ here is -3 which is odd.

$\therefore$ No values of A and B exist as solutions.

4.70:

$P(p+1)+q(q+1) = n(n+1) \Rightarrow p(p+1) = (n-q)(n+q+1)$.

This means that 'p' divides $(n-q)$ or $(n+q+1)$. (A)

If p divides $(n-9)$, then $n-q \geq p$.

$\therefore n+q+1 = n-q+2q+1 > n-q+1 \geq p+1$. (B)

But this will contradict

$p(p+1)+q(q+1) = (n-q)(n+q+1)$.

So p has to divide $(n+q+1)$.

This means that $(n+q+1) = kp$ for since positive integer k and $p+1 = k(n-q)$. (C)

We note that $(p+q)(p+q+1) = p(p+1)+q(q+1)+2pq > n(n+1) \Rightarrow p+q > n$. (D)

Without loss of generality, we may assume that $p \geq q$.

As $n(n+1) > p(p+1)$, we get $n > p$.
Now $n+q+1 = kp > n > p \rightarrow k > 1$.

Also $kp = n+q+1 < (p+q)+q+1 \leq 3p+1 < 4p \Rightarrow k < 4$.

$\therefore K$ has to be 2 or 3.

Suppose $k = 2$. Then $n = 2p-q-1$ and $p+1 = 2n-2q$.

This implies that $2n = 4p-2q-2 = p+1+2q$ and therefore $3p-3 = 4p$.

Hence 3 divides the prime number q which implies that $q = 3$. This gives $p = 5$ and $n = 6$. When $k = 3$, we get $n = 3p-q-1$ and $p+1 = 3n-3q$. This gives

$3n = 9p - 3q - 3 = p + 1 + 3q$ and therefore $8p - 4 = 6q$.

This gives $3n = 9p - 3q - 3 = p + 1 + 3q$ and therefore $8p - 4 = 6q$. This implies that $q = 2, p = 2, n = 3$.

Thus the solutions are given by $(p, q, n) = (3, 5, 6)$ or (5,3,6) or(2,2,3).

4.71: Observe that

$x^2 - 3xy + 2y^2 + x - y = (x - y)(x - 2y + 1)$.

Thus 17 divides either $x - y$ or $x - 2y + 1$. Suppose that 17 divides $(x - y)$. In this case $x \equiv y(\text{mod}17)$ and hence,

$$\begin{aligned} x^2 - 2xy + y^2 - 5y + 7y &\equiv y^2 - 2y^2 + y^2 - 5y + 7y \\ &\equiv 2y(\text{mod}19). \end{aligned}$$

Thus the given condition that 17 divides $x^2 - 2xy + y^2 - 5y + 7y$ implies that 17 also divides $2y$ and hence y itself.

But then $x \equiv y(\text{mod}17)$ implies 17 divides x also.

Hence in this case 17 divides $xy - 12x + 15y$.

Suppose on the other hand that 17 divides $x - 2y + 1$. Thus $x \equiv 2y - 1(\text{mod}17)$ and hence

$$x^2 - 2xy + y^2 - 5x + 7y \equiv y^2 - 5y + 6(\text{mod}17).$$

Thus 17 divides $y^2 - 5y + 6$; But $x \equiv 2y - 1(\text{mod}17)$ also implies that

$$xy - 12x + 15y \equiv 2(y^2 - 5y + 6)(\text{mod}17).$$

Since 17 divides $y^2 - 5y + 6$, it follows that 17 divides $xy - 12x + 15y$.

4.72: Putting $a-1=p, b-1=q$ and $c-1=r$, the equation may be written in the form,

$$pqr = 2(p+q+r)+4$$

where p, q, n are integers such that $0 \leq p \leq q \leq r$. Observe that $p=0$ is not possible, for then $0 = 2(p+q)+4$ which is impossible in non-negative integers. Thus we may write this in the form,

$$2\left(\frac{1}{pq}+\frac{1}{qr}+\frac{1}{rp}\right)+\frac{4}{pqr}=1.$$

If $p \geq 3$, then $q \geq 3$ and $r \geq 3$. Then left side is bounded by $6/9+4/27$ which is less than 1. We conclude that $p=1$ or 2.

case 1: Suppose $p=1$; Then we have $qr = 2(q+r)+6$ or $(q-2)(r-2)=10$. This fixes $q-2=1, r-2=10$ or $q-2=2$ and $r-2=5 (q \leq r)$. This implies $(p,q,r) = (1,3,12), (1,4,7)$.

Case 2: If $p=2$, the equation reduces to $2qr = 2(2+q+r)+4$ or $qr = q+r+4$. This reduces to $(q-1)(r-1)=5$. Hence $q-1=1$ and $r-1=5$ is the only solution. This gives $(p,q,r)=(2,2,6)$. Thus we have the triples (a,b,c) as $(2,4,13)$, $(2,5,8)$, $(3,3,7)$.

4.73: If the number $abcdef$ is lucky, then so $999999 - abcdef$. Thus the lucky numbers can be paired so that the sum of each pair is 999999. The sum of all lucky numbers is therefore a multiple of 999999.13 divides 999999.

4.74: Suppose that the person was born in the year $19xy$ (where $1, 9, x, y$ are digits). His age in the year 1996 is
$1996 - 19xy = 96 - 10x - y$. (A)

If this equals the sum of digits of the year in which he was born, then $96 - 10x - y = 1 + 9 + x + y$. (B)

i.e., $86 = 11x + 2y$. (C)

Now $11x + 2y = 86$ has no solution with $0 \leq x, y \leq 9$. Also we fix that $11x + 2y = 75$ also has no solution but $11x + 2y = k$ has solution, when $76 \leq k \leq 85$. (D)

Thus the last year previous to 1996, which has the same property is 1985.

4.75: Since the sum is k, the centre square must have $k - (33 + 31) = k - 64$. (A)

Again the square at the 3rd row, 3rd column must have solution $k - 59$. (B)

Thus the second sequence in the 3rd column must have

$k - (33 + k - 59) = 26$. (C)

$k - 69$	36	33
38	$k - 64$	26
31	28	$k - 59$

Second square on the first column has

$k - (k - 64 + 26) = 38$. (D)

The first square in the first column has

$k - (38 + 31) = k - 69$. (E)

The second square in the first column has

$k - (k - 69 + 33) = 36$. (F)

Using the main diagonal, we have $3k - 69 - 64 - 59 = k$ i.e., $k = 96$.

Thus the magic square is

27	36	33
38	32	26
31	28	37

4.76: In the sequence 101, 201, 301, $\cdots$ 901, 1001, 1101, $\cdots$ 1901, 2001, there are $\dfrac{2001 - 101}{100} + 1 = 20$ terms.

The number of even numbers in these terms are alternatively 1 and 2 up to 1901 and for 2001 it is 3.

Thus $E(101) \times E(201) \times \cdots E(2001)$

$= 1 \times 2 \times 1 \times 2 \times \cdots \times 1 \times 3$

$= 2^9 \times 1^{10} \times 3 = 1536$

$E[E(101) \times E(102) \times - E(2001)] \; = E(1536) = 1.$

★★★★★

UNIT 4: INEQUALITIES

5.01: For real numbers a, b, x, y, we have the inequality

$$ax + by \le \sqrt{a^2 + b^2}\sqrt{x^2 + y^2}.$$

Hence $60 = 5x + 12y \le \sqrt{5^2 + 12^2}\sqrt{x^2 + y^2}$

$$= 13\sqrt{x^2 + y^2}.$$

$\therefore$ The minimum value of $\sqrt{x^2 + y^2}$ is $60/13$.

(It has to be noted that $\min \sqrt{x^2 + y^2}$ for x, y satisfying $5x + 12y = 60$, is the perpendicular distance of $(0, 0)$ from the line $5x + 12y = 60$).

5.02: $2 \le |x - 1|$ is equivalent to the inequalities $2 \le x - 1$ and $x \ge 1$ or $2 \le 1 - x$ and $x \le 1$. Thus $x \ge 3$ or $x \le -1$.

Similarly $|x - 1| \le 5$ is equivalent $x \ge 1$ and $x - 1 \le 5$ or $x \le 1$ and $1 - x \le 5$ and hence $1 \le x \le 6$ or $-4 \le x \le 1$.

Thus $2 \le |x - 1| \le 5$ is equivalent to $3 \le x \le 6$ or $-4 \le x \le -1$.

5.03:

$$\begin{aligned}\frac{4x^2 + 8x + 13}{6(x + 1)} &= \frac{4(x^2 + 2x + 1) + 9}{6(x + 1)} \\ &= \frac{4(x + 1)^2 + 9}{6(x + 1)} \\ &= \frac{2}{3}(x + 1) + \frac{9}{6(x + 1)}.\end{aligned}$$

For positive $a, b; a+b \geq 2\sqrt{ab}(AM \geq GM)$.

$$\therefore \frac{2}{3}(x+1) + \frac{9}{6(x+1)} \geq 2\sqrt{\frac{2}{3} \cdot \frac{9}{6}} = 2.$$

$\therefore$ The smallest value of the expression is 2.

5.04: $\sqrt[3]{x+9} - \sqrt[3]{x-9} = 3$.

Cubing both sides,

$(x+9)-(x-9)-3\sqrt[3]{x+9}\cdot\sqrt[3]{x-9}(\sqrt[3]{x+9}-\sqrt[3]{x-9}) = 27$.

i.e., $-\sqrt[3]{x^2-81} = (\sqrt[3]{x+9} - \sqrt[3]{x-9}) = 3$.

$\Rightarrow -\sqrt[3]{x^2-81} = 1 \quad \Rightarrow x^2 = 80$.

Thus x^2 lies between 75 and 85 as it is 80.

5.05:

$$\begin{aligned} \left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) &\geq 9 \\ \Rightarrow \left(\frac{x+1}{x}\right)\left(\frac{y+1}{y}\right) &\geq 9 \\ \Rightarrow \quad xy+x+y+1 &\geq 9xy \end{aligned}$$

$\Rightarrow 2 \geq 8xy \quad \Rightarrow xy \leq 1/4$. (A)

By $A.M - G.M$ inequality,

$\sqrt{xy} \leq \frac{x+y}{2} = \frac{1}{2} \quad \therefore xy \leq 1/4$ by squaring. (B)

Thus $(A) = (B)$; i.e., $\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) \geq 9$.

5.06: Let a, b, c, d be the weights in kg of the four bags where $a \leq b \leq c \leq d$. Then $103, 105, \cdots 109$ **are**

$a+b, a+c, \cdots c+d$ in some order. We have $a+b \le a+c \le a+d \le b+d \le c+d$ and $a+b \le a+c \le c+b \le b+d \le c+d$.

So $(a+b)$ is least and $(c+d)$ is greatest.

Of the remaining four, the least is $(a+c)$ and the greatest is $(b+d)$.

It follows that $a+b = 103$, $c+d = 109$, $a+c = 105$, $b+d = 107$, $a+d = 106 = b+c$

$$\begin{aligned} \text{Hence } 2a &= (a+b)+(a+c)-(b+c) \\ &= 103+105-106 = 102 \\ \therefore a &= 51. \end{aligned}$$

$\therefore$ The weight of the lightest bag is 51 kg.

5.07: The expression in the first bracket is equal to

$$\begin{aligned} &\frac{\sqrt{1+a}}{\sqrt{1+a}-\sqrt{1-a}} + \frac{\sqrt{1-a}}{\sqrt{1+a}-\sqrt{1-a}} \\ &= \frac{(\sqrt{1+a}+\sqrt{1-a})^2}{(1+a)-(1-a)} \\ &= \frac{1+a+1-a+2\sqrt{1-a^2}}{2a} \\ &= \frac{1+\sqrt{1-a^2}}{a}. \qquad \text{(A)} \end{aligned}$$

Thus the given expression is equal to

$$\left\{\frac{1+\sqrt{1-a^2}}{a}\right\} \times \left\{\sqrt{\frac{1-a^2}{a^2}} - \frac{1}{a}\right\} = \frac{(1-a^2)-1}{a^2} = -1.$$

Thus the expression lies between -2 and $+2$.

5.08: Let $a_1, a_2 \cdots a_7$ be the amounts in rupees with the seven persons. Suppose that any three of them possess a

total amount less than Rs.142. Then $a_i + a_y + a_k < 142$ for distinct i, j, k. There are $7C_3 = 35$ such inequalities. Add them, since each a_i is repeated $6C_2 = 15$ times, we get

$$\begin{aligned}
15(a_1 + a_2 + \cdots + a_7) &< 35 \times 142 \\
\therefore a_1 + a_2 + \cdots + a_7 &< \frac{35 \times 142}{15} = 331 + 1/3 \\
\text{So if } a_1 + a_2 + \cdots + a_7 &= 332, \\
\text{then} a_i + a_j + a_k &\geq 142 \text{ for some triple } (a_i, a_r, a_k).
\end{aligned}$$

Suppose that 6 persons have Rs.47.5 each and one person Rs.47. Then the total is s 332 and so three of them possess a total of Rs.143 (move) So, the desired value of n is 142.

5.09: Let the cost of one chocolate be x paise and the cost of one cup of coffee be y paise. Then, if we assume that it is possible to buy a cup of coffee and 3 chocolates for a rupee, by the hypothesis of the problem, we have the in equalities.

$$125y < 175x < 126y \tag{1}$$

$$y + 3x \leq 100 \tag{2}$$

(1) gives $125y - 125y < 175x - 125y < 126y - 125y$

i.e., $0 < 175x - 125y < y$

$$\Rightarrow 0 < 25(7x - 5y) < y. \tag{3}$$

Now x and y are positive integers and also $7x - 5y > 0$

$$7x - 5y = 1 \rightarrow x = \frac{1 + 5y}{7}$$

is an integer and $y > 25$ by (3).

The least value of y satisfying this is $y = 32$ and in this case $x = 23$. (4)

But $x = 23, y = 32$ gives $y + 3x = 101 > 100$ (5)
contradicting (2)

$\therefore$ A larger value of y gives a larger value of $x = \dfrac{1+5y}{7}$.

If $7x - 5y \geq 2$, then $y > 50$ by (3) and hence
$7x \geq 5y + 2 > 252$. (6)

Thus $x > 36$ and $y > 50$ contradicting (2). This shows that one rupee is not enough to buy one cup of coffee and three chocolates.

5.10: As a, b, c are the sides of a triangle, we have a, b, c are all > 0 and also $c + a - b, a + b - c, b + c - a$ are all > 0. Now by $AM - GM$ inequality.

$$\frac{a}{c+a-b} + \frac{b}{a+b-c} + \frac{c}{b+c-a} \geq$$

$$3\left[\frac{abc}{(c+a-b)(a+b-c)(b+c-a)}\right]^{\frac{1}{3}}. \qquad (1)$$

Now we have $a^2 \geq a^2 - (b-c)^2$.

$\Rightarrow a^2 \geq (a+b-c)(a-b+c)$. (2)

Similarly $b^2 \geq (b+c-a)(b-c+a)$ (3)

and $c^2 \geq (c+a-b)(c-a+b)$. (4)

Multiplying both sides of inequalities of (2), (3), (4),
$a^2b^2c^2 \geq (a+b-c)^2(b+c-a)^2(c+a-b)^2$. (5)

Taking the positive sq root of (5),

we get $abc \geq (a+b-c)(b+c-a)(c+a-b)$. (6)

i.e., $\dfrac{abc}{(a+b-c)(b+c-a)(c+a-b)} \geq 1$.

$$\Rightarrow [\frac{abc}{(a+b-c)(b+c-a)(c+a-b)}]^{\frac{1}{3}} \geq 1.$$

So (1) implies that $\dfrac{a}{c+a-b} + \dfrac{b}{a+b-c} + \dfrac{c}{b+c-a} \geq 3$.

5.11: $n = \dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{3+\sqrt{5}}} < \dfrac{\sqrt{5}+\sqrt{5}}{\sqrt{3+\sqrt{4}}} = \dfrac{2\sqrt{5}}{\sqrt{5}} = 2$. (A)

Also $n = \dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{3+\sqrt{5}}} > \dfrac{\sqrt{3}+\sqrt{3}}{\sqrt{3+\sqrt{9}}} = \dfrac{2\sqrt{3}}{\sqrt{6}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$. (B)

Thus $\sqrt{2} < r < 2$.

5.12: If x be the number of sides of the polygon, sum of the interior angles $= (2n-4)$ at angles.

$\therefore (2n-4) < 2 \times (5) + (n-5)$ (right angles) i.e., $n < 9$.

$\therefore$ Maximum number of sides $= 8$.

5.13: Let the angles be $2x, 3x, 4x$ and $Q = Kx$ (say). Now $(9+k)x = 360°$.

When $K = 1, 0 < Q < 90°$.

When $K = 3, Q = 90°$.

When $k = 6, 90° < Q < 180°$.

$\therefore Q$ cannot be exactly determined as it varies with k. Thus the statement is not true always.

5.14: $a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca)$
$= 1 - 2(ab+bc+ca)$. (1)

So, it is enough to prove that

$$1-2(ab+bc+ca)\geq 4(ab+bc+ca)-1$$

i.e., $6(ab+bc+ca)\leq 2$ or $3(ab+bc+ca)\leq 1$. (2)

This happens if and only if

$$\begin{aligned}0 &\leq 1-3(ab+bc+ca)\\ &=(a+b+c)^2-3(ab+bc+ca)\\ &=(a-b)^2+(b-c)^2+(c-a)^2\end{aligned}$$

which is always true. Clearly equality holds when $a=b=c=1/3$.

5.15: Let $M=$ Number of men.

$x=$ Number of flowers each man got.

$$Mx+(12-M)(x+2)=142$$

$$\therefore 6x-M=59$$

$$\therefore x=9+\frac{5+M}{6} \tag{1}$$

M,x are whole numbers;
then $M>$ no of women i.e., $6<M<12$ (2)

$$\therefore M=7$$

by (1) and $x=11$. $\therefore$ Each woman got $x+2=11+2=13$ flowers.

5.16: $x,y,z\geq 0, x+y+z=11$.

$$\begin{aligned}xyz+xy+yz &= (x+1)(y+1)(z+1)-(x+y+z)-1\\ &= pqr-12 \text{ and } p+q+r=14\\ \text{where} \quad & p=x+1, q=y+1, r=z+1\end{aligned}$$

pqr is maximum subject to $p + q + r = 14$, when $p = q = r = 14/3$.

$$\therefore \text{ regd maximum } = \frac{14 \times 14 \times 14}{3 \times 3 \times 3} - 12$$

$$= \frac{2744 - 324}{27} = 89\frac{17}{27}.$$

If x, y, z are restricted to be integers, p, q, r are integers. Their value for maximum of $pqr - 12$ when $p+q+r = 14$ should be 5,5 and 4. $\therefore$ max.value $= 100 - 12 = 88$.

5.17: The bill for triplets and niece is Rs.(160-75)=85. If the age in completed years of triplets is x and that of her niece is y, $5(3x + y) = 85$ and $x > y \Rightarrow 3x + y = 17$.

Clearly $x \leq 5$. If $x = 5, 3 \times 5 + 2 = 17$ and $5 > y = 2$ but for $x = 4$, we get $3 \times 4 + 5 = 17$ and $3 < y = 5$.

Thus for $x \leq 4$, the age of the niece is not less that of the triplets. So the age of the triplets is 5.

5.18: Assume that $a^2 + b^2 + c^2 < \sqrt{3}abc$.

Then by $AM - GM$ inequality.

$$3\sqrt[3]{a^2b^2c^2} \leq a^2 + b^2 + c^2 < \sqrt{3}abc$$

$\therefore$ It follows that $abc > 3\sqrt{3}$. (A)

Also by the given condition on a, b, c.

$$\frac{a^2b^2c^2}{3} \leq \frac{(a+b+c)^2}{3} \leq a^2 + b^2 + c^2 < \sqrt{3}abc$$

$\Rightarrow abc < 3\sqrt{3}$. (B)

(A) and (B) contradict $\therefore a^2 + b^2 + c^2 \geq \sqrt{3}abc$.

5.19: $x^2(y-z)+y^2(z-x)+z^2(x-y)$

$$\begin{aligned} &= x^2(y-z)+y^2z-y^2x+z^2x-z^2y \\ &= x^2(y-z)+(y^2z-z^2y)+(z^2x-y^2x) \\ &= x^2(y-z)+yz(y-z)+x(z+y)(z-y) \\ &= (y-z)[x^2-xy-xz+yz] \\ &= (y-z)(x-y)(x-z). \qquad \text{(A)} \end{aligned}$$

Now as x, y, z are the sides of a triangle and by the triangle inequality, $|y-z|<x, |x-y|<z; |x-z|<y$.

$\Rightarrow |y-z|\ |x-y|\ |x-z| \ < xyz$. (B)

Hence $|x^2(y-z)+y^2(z-x)+z^2(x-y)|$

$= |(y-z)(x-y)(x-z)|$

$= (x-y)(y-z)(x-z) < xyz$. (C)

5.20: By the $AM-GM$ inequality,

$$xy \leq (\frac{x+y}{2})^2 = 1 \qquad \text{(A)}$$

$$\begin{aligned} x^3y^3(x^3+y^3) &= (xy)^3(x+y)(x^2+y^2-xy) \\ &= 2(xy)^3\{(x+y)^2-3xy\} \\ &= 2(xy)^3(4-3xy). \qquad \text{(B)} \end{aligned}$$

$\therefore$ It is enough to prove that $(xy)^3(4-3xy) \leq 1$. (C)

Let $xy = t$: Coincides the four positive quantities $4, -3t, t, t, t$.

But $AM-GM$ inequality $t^3(4-3t) \leq \left(\dfrac{4-3t+3t}{4}\right)^4 = 1$.

5.21: If the convex polygon has x sides, we are given that, the sum of the angles of the polygon except one $= 2190^\circ = 12 \times 180^\circ + 30^\circ$; Last angle x° is such that $0^\circ < x^\circ < 180^\circ$.

The sum of all the x interior angles is $(2n - 4) \times 90^\circ$.

Thus $2190 + 0^\circ < (n-2) \times 180 < 2190 + 180^\circ$.

i.e., $\dfrac{2190}{180} < (n-2) < \dfrac{2370}{180}$

i.e., $n - 2 = 13 \quad \Rightarrow n = 15$.

Hence the given polygon has 15 sides.

5.22: From the given equation,
we get $(x-a)^2 = 3 - a$ or $x = a \pm \sqrt{3-a}$. (A)

As the roots are real, $3 - a \geq 0$ or $a \leq 3$. (B)

As the roots are also less than 3,

we have $a + \sqrt{3-a} < 3$ and $a - \sqrt{3-a} < 3$ (C)

From $a + \sqrt{3-a} < 3$,
we get $\sqrt{3-a} < 3 - a$. (D)

This implies $a \neq 3$. (E)

Hence $a < 3$. (F)

As $(3-a)$ can be written as $\sqrt{3-a}\sqrt{3-a}$
and $\sqrt{3-a} > 0$.
We get $1 < \sqrt{3-a}$. (G)

Hence a is not greater than or equal to 2. (H)

in other words, $a < 2$ (I)

When $a < 2, a - \sqrt{3-a} < 3$ is certainly true. (J)

5.23: It is enough to prove that

$$\frac{a}{b} + 3\sqrt[3]{a/b} + 3\sqrt[3]{b/a} + \frac{b}{a} \leq 2(a+b)(\frac{1}{a} + \frac{1}{b}).$$

This is equivalent to proving

$$3(\sqrt[3]{a/b} + \sqrt[3]{b/a}) \leq 4 + (a/b + b/a).$$

Let $x = 3\sqrt[3]{a/b}$. Then x is a positive number,

$$\therefore 3(x + x^{-1}) \leq 4 + x^3 + x^{-3}$$

$$\text{i.e, } x^6 - 3x^4 + 4x^3 - 3x^2 + 1 \geq 0$$

$$\text{i.e., } (x-1)^2(x^4 + 2x^3 + 2x + 1) \geq 0.$$

This inequality is clearly true for $x \leq 0$. This inequality becomes equality if $x = 1$ i.e., $a = b$.

Second Proof: Apply $AM - GM$ inequality to $1, 1, a/b$ we get

$$\sqrt[3]{1.1.a/b} \leq \frac{1+1+a/b}{3}$$

$$\text{i.e., } 3\sqrt[3]{a/b} \leq 2 + a/b.$$

Similarly we get $3\sqrt[3]{b/a} \leq 2 + b/a$

$$3\sqrt[3]{a/b} + 3\sqrt[3]{b/a} \leq 4 + a/b + b/a.$$

Equality occurs if $1 = a/b$ i.e., if $a = b$.

5.24: $108 = 3^3.4$

$192 = 4^3.3$

$500 = 5^3.4$

$1080 = 6^3.5$.

This gives $x = \log_3 108 = 3 + \log_3 4$

$y = \log_4 192 = 3 + \log_4 3$

$z = \log_5 500 = 3 + \log_5 4$

$w = \log_6 1080 = 3 + \log_6 5$

$4 = 3^{x-3}; 3 = 4^{y-3}; 5 = 6^{w-3}$

$3.5 = 15 = 4^{y-3} \cdot 4^{\frac{1}{z-3}} = 4^{y-3} + \frac{1}{y-3}$

$4.6 = 24 = 5^{z-3} \cdot 5^{\frac{1}{w-3}} = 5^{z-3} + \frac{1}{z-3}$

$\therefore y - 3 + \frac{1}{z-3} < 2 < (z-3) + \frac{1}{z-3}$

$z - 3 + \frac{1}{w-3} < 2(w-3) + \frac{1}{w-3}$

$x - 3 > 1$ and $0 < w - 3 < 1$

$\therefore y - 3, \ \angle z - 3 < w - 3 < x - 3$

i.e., $y < z < w < x$

i.e., $\log_4^{192} < \log_5^{500} < \log_6^{1080} < \log_3^{108}$.

5.25: The harmonic mean of x, y, z is given by

$h = \left(\frac{1}{z}\Sigma\frac{1}{x}\right)^{-1}$ and the harmonic mean of

$1+x, 1+y, 1+\ z$ i.e,$H = \left(\frac{1}{3} \in \frac{1}{1+x}\right)^{-1}$. (A)

Let $a = \frac{1}{x}; b = \frac{1}{y}; c = \frac{1}{z}$. Then assuming $x \geq y \geq z$,

$$\frac{a}{1+a} + \frac{b}{1+b} \leq \frac{a}{1+b} + \frac{b}{1+a}. \quad \text{(B)}$$

$$\frac{b}{1+b} + \frac{c}{1+c} \leq \frac{b}{1+c} + \frac{c}{1+b}. \quad \text{(C)}$$

$$\frac{c}{1+c} + \frac{a}{1+a} \leq \frac{c}{1+a} + \frac{a}{1+c}. \quad \text{(D)}$$

Adding $2\epsilon(\frac{a}{1+a}) \leq \Sigma\frac{b+c}{1+a}$. (E)

$\Rightarrow 3\Sigma(\frac{a}{1+a}) \leq \Sigma\frac{a+b+c}{1+a}$. (F)

Since $\frac{a}{1+a} = \frac{1}{x+1}, \frac{1}{1+a} = \frac{x}{x+1} = 1 - \frac{1}{x+1}$. (G)

$\because 3\Sigma(\frac{1}{1+x}) \leq \Sigma(1 - \frac{1}{x+1})$

$= \Sigma(3 - \Sigma\frac{1}{x+1})$. (H)

$\Rightarrow \frac{1}{3}\Sigma\frac{1}{1+x} \leq \frac{1}{3}\Sigma\frac{1}{x}(1 - \frac{1}{3}\Sigma\frac{1}{x+1})$. (I)

$\Rightarrow \frac{1}{H} \leq \frac{1}{h}(1 - \frac{1}{H})$. (J)

$\Rightarrow 1 + h \leq H$.

5.26: The given condition means that $\frac{abc}{2\Delta} \leq \frac{\Delta}{s-a}$.

This gives $abc \leq 2s(s-b)(s-c)$.

If we write $s-a = x, s-b = y, s-c = z$, We have $y+z = a, z+x = b, x+y = c, x+y+z = s$.

The given condition can be written as $(y+z)(z+x)(x+y) \leq 2(x+y+z)yz$

$\therefore xy(x+y) + zx(x+x) \leq yz(y+z)$. (A)

Suppose $a \leq b$. then $y+z \leq z+x$ and hence $y \leq x$.

$\therefore$ (A) implies $xy(x+y) \leq 0$ which is not true as x, y, z are positive.

Similarly we observe that $a > c$. This proves the first part.

Second Part: Suppose $2R \leq r_b$. Proceeding as before, $yx(y+x) + yz(y+z) \leq xz(x+z)$. (B)

Adding this with (A) $2xy(x+y) \leq 0$ (C) which is again not true. $2R > r_b$.. (D)

Similarly we prove that $2R > r_c$. (E)

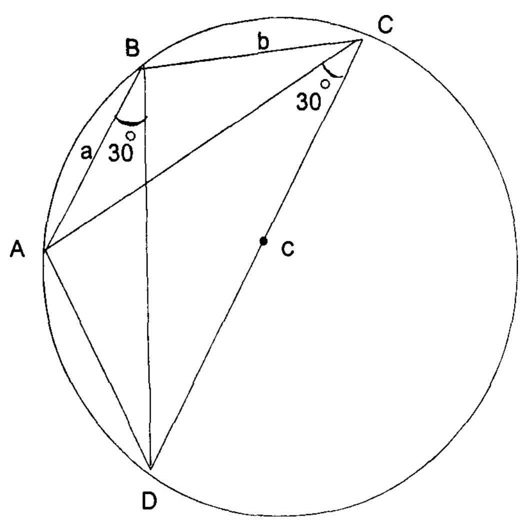
5.27: Applying cosine rule to $\triangle ABC$,

$AC^2 = a^2 + b^2 - 2ab\cos 120° = a^2 + b^2 + ab$

$\angle DAC = \angle DBC = 120° - 30° = 90°$.

Therefore $c^2 = \frac{AC^2}{\cos^2 30°} = \frac{4}{3}(a^2 + b^2 + ab)$.

So, $c^2 - (a+b)^2 \doteq \frac{4}{3}(a^2 + b^2 + ab) - (a^2 + b^2 + 2ab)$.



This proves $c \geq a + b$ and thus (i) is true. For proving (ii), consider the product.

$\theta = (\alpha + \beta + \gamma), (\alpha - \beta - \gamma)(\alpha + \beta - \gamma)(\alpha - \beta + \gamma)$

where $\alpha = \sqrt{c+a}, \beta = \sqrt{c+b}$ and $\gamma = \sqrt{c-a-b}$.

Expanding the product, we get

$$\theta = (c+a)^2+(c+b)^2+(c-a-b)^2-2(c+a)(c+b)-2(c+a)(c-a-b)$$

$$-2(c+b)(c-a-b)$$

$$= -3c^2 + 4c^2 + 4b^2 + 4ab = 0.$$

Thus at least one of the factors must be equal to 0. Since $\alpha+\beta+r > 0$ and $\alpha+\beta-r > 0$, it follows that the product of the remaining two factors is 0. This gives $\sqrt{c+a} - \sqrt{c+b} = \sqrt{c-a-b}$ or $\sqrt{c+a} - \sqrt{c+b} = -\sqrt{c-a-b}$.

We conclude that $|\sqrt{c+a} - \sqrt{c+b}| = \sqrt{c-a-b}$.

5.28: Using $|a-b| \geq |c|$, we get $(a-b)^2 \geq c^2$ which is equivalent to $(a-b-c)(a-b+c) \geq 0$.

Similarly $(b-c-a)(b-c+a) \geq 0$

and $(c-a-b)(c-a+b) \geq 0$.

Multiplying these inequalities, we get

$$-(a+b-c)^2(b+c-a)^2(c+a-b)^2 \geq 0$$

This forces the product of the three squares on the L.H.S equal to 0.

Hence it follows that either $a+b=c$ or $b+c=a$ or $c+a=b$.

5.29: Suppose the equation $x^2+x+4\lambda = 0$ has no real roots.

Then $1-16\lambda < 0$. (A)

We note that $1-16(a^3+a^2bc) < 0$

$\Rightarrow 1-16a^2(a+bc) < 0$

$\Rightarrow 1-16a^2(1-b-c+bc) < 0$

$\Rightarrow 1 - 16a^2(1-b)(1-c) < 0$

$\Rightarrow \frac{1}{16} < a^2(1-b)(1-c)$. (B)

Similarly we get $\frac{1}{16} < b^2(1-c)(1-a)$ (C)

and $\frac{1}{16} < c^2(1-a)(1-b)$. (D)

Multiplying these inequalities (B),(C),(D)

$a^2b^2c^2(1-a)^2(1-b)^2(1-c)^2 > \frac{1}{16^3}$. (E)

However, $0 < a < 1$ implies that $a(1-a) \leq 1/4$. (F)

Hence

$a^2b^2c^2(1-a)^2(1-b)^2(1-c)^2 = \{(a(1-a)^2)(b(1-b)^2)(c(1-c)^2)\}$
$\leq \frac{1}{16^3}$; a contradiction.

$\therefore$ we conclude that the give equation has real roots.

5.30: Required to prove

$$\frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} \geq \frac{9}{a+b+c}. \tag{A}$$

$$\Longleftrightarrow \frac{a^3+b^3+c^3}{a^2b^2c^2} \geq \frac{9}{a+b+c}. \tag{B}$$

$$\Longleftrightarrow (a+b+c)(a^3+b^3+c^3) \geq 9a^2b^2c^2 = (3abc)^2. \tag{C}$$

$$\Longleftrightarrow (a+b+c)(a^3+b^3+c^3) \geq (a^2+b^2+c^2)^2. \tag{D}$$

It is enough that we prove (D) to prove (A).

By cauchy-Schwartz inequality, we get

$$\left[(\sqrt{a})^2 + (\sqrt{b}^2) + (\sqrt{c})^2\right] \cdot \left[(a\sqrt{a})^2 + (b\sqrt{b})^2 + (c\sqrt{c})^2\right]$$
$$\geq \left(\sqrt{a}(a\sqrt{a}) + \sqrt{b}(b\sqrt{b}) + \sqrt{c}(c\sqrt{c})\right)^2$$

i.e., $(a+b+c)(a^3+b^3+c^3) \geq (a^2+b^2+c^2)^2$.

Thus (A) is proved.

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